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Computation of the Picard Group of the Moduli Stack of Elliptic Curves

Tesi di Laurea Magistrale

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Introduction

The purpose of this thesis is to give an exposition of the computation of the Picard group of the moduli stack of elliptic curves.

Let $\mathcal{M}_{1,1,S}$ denote the fibered category over (Sch/S), whose objects are collections of data $(T,(E,e))$, where T is a S-scheme and (E,e) is an elliptic curve over T, and whose maps are suitable morphisms. $\mathcal{M}_{1,1,S}$ is an algebraic stack, with respect to the fpqc topology.

In 1965 David Mumford proved that the Picard group of the moduli space $\mathcal{M}_{1,1,k}$ is equal to $\mathbb{Z}/12\mathbb{Z}$, where k is a field of characteristic not 2 or 3 [Mum65].

In 2010 William Fulton and Martin Olsson supplemented the previous exceptional work of Mumford with the article "The Picard group of $\mathcal{M}_{1,1}$" [FO10]. The work of the thesis is completely inspired by this article. The authors have the double merit of having revisited Mumford's previous results in the more modern stack language and of having generalized the proof to a general base scheme S. In particular, the main theorem proved is the following.

Theorem 0.0.1. Let S be a connected scheme. Then the map

$$\begin{gathered}\mathbb{Z}/12\mathbb{Z} \times \mathrm{Pic}(\mathbb{A}^1_S) \longrightarrow \mathrm{Pic}(\mathcal{M}_{1,1,S}) \\ (i,\mathcal{L}) \mapsto \lambda^{\otimes i} \otimes p^*\mathcal{L}\end{gathered} \tag{1}$$

is an isomorphism if either of the following hold:

(i) S is a $\mathbb{Z}[1/2]$-scheme.

(ii) S is reduced.

The line bundle λ is the well-known Hodge bundle. The morphism

$$\rho\colon \mathcal{M}_{1,1,S} \to \mathbb{A}^1_S$$

is the map induced by the j-invariant which makes $\mathbb{A}^1_S$ into the coarse moduli space of the stack $\mathcal{M}_{1,1,S}$.

The outline of this thesis is as follows. In chapter 1 we recall some basic notions in algebraic geometry, such as Grothendieck topologies, fiber categories and the fundamental tool of *descent*. Only after doing this, we describe the objects we will work with in this thesis, that is the algebraic stacks, and we present their main properties. At the end of the chapter, we focus on the notion of quasi-coherent sheaf on a stack. We are in particular interested in the line bundles. At first we give a definition which involves the *lisse-étale* site of an algebraic stack. While this concept is elegant, it is difficult to work with in practice, so we provide an equivalent definition for the case of a quotient stacks. This relates the notion of line bundle on a stack with that of G-equivariant sheaf over a scheme, with G a smooth group scheme. This is very useful for our purposes since $\mathcal{M}_{1,1,S}$ is a quotient stack. The theme of chapter 2 is that of elliptic curves. We start recalling some basic concepts of the classical theory of elliptic curves over a field k, and after we treat more in general the problem of families of curves parameterized by a generic scheme T. We describe in detail $\mathcal{M}_{1,1}$. It is a fibered category over (Sch) and we prove that it is a separated Deligne-Mumford stack, with respect to the fpqc topology. Moreover, we prove that it is a quotient stack and we describe the morphism into its coarse moduli space $\mathbb{A}^1$. We also describe $\mathcal{M}_{1,1,S} := \mathcal{M}_{1,1} \times_{Spec(\mathbb{Z})} S$ for a general scheme S and some of its properties. After these preliminaries, we are ready to explain the content of the article of Fulton and Olsson. We do it in chapter 3. The proof is divided into many steps, in each of which the problem is analyzed by restricting oneself to a specific type of base scheme S for $\mathcal{M}_{1,1,S}$, and we study them in detail. First of all, we give a proof of the main Theorem in the case of a $\mathbb{Z}[1/6]$-scheme S. This particular situation is very enlightening because it emphasizes the close relation between the Hodge bundle λ and the Picard group of the coarse moduli space of $\mathcal{M}_{1,1,S}$. Afterwards, we prove the Theorem in the case of S normal. Here the point of view is different from the previous case: it involves the specific nature of the Weil divisors of the scheme $\mathbb{A}^1_S$ and the structure of the line bundles over a classifying stack $\mathbb{B}G$. From this situation it is possible to go back to prove the thesis when S is a reduced scheme. Afterwards, modifying this with arguments coming from deformation theory, we prove the Theorem for a $\mathbb{Z}[1/2]$-scheme S. Finally, we exhibit an example of a non reduced k-scheme S, with characteristic of k equal to 2, in which case the thesis of the Theorem fails.

CHAPTER 1

Preliminary Notions

Moduli problems are of crucial interest in Algebraic Geometry. Approximately, the study of moduli is the study of families of some geometric objects with some specific additional conditions. For example, one would like to study the family $\mathcal{M}_{1,1}$ of all elliptic curves with suitable morphisms between them (this is what Chapter 2 is devoted to). What we want to try to do is look for a scheme M, which is the total space of the parameters, and a universal family $\mathcal{E}$ on M such that every other family is obtained through a pull back of this one. By Yoneda, this is the same of saying that the functor

$$\begin{aligned} (Sch)^{op} &\to (Set) \\ X &\mapsto \{ \textit{ Family of elliptic curves over } X\}/\cong \end{aligned}$$

is representable, i.e. it is isomorphic to $Hom_{(Sch)}(\bullet, X)$. Unfortunately, this often does not happen for a number of reasons, which in a certain sense can usually be traced back to the presence of non-trivial automorphisms of the objects of the family. The problem of classification can be solved by introducing a new geometric concept: stacks, and more specifically, algebraic stacks.

The objective of this first chapter is to recall the basic theoretical notions necessary to define and develop the theory on stacks.

First of all we define some kind of morphisms which are fundamental since allow to extend globally some local properties. Locally doesn't mean Zariski locally, but (often) locally in the fpqc topology. It is not a topology in the common sense, but it is a Grothendieck topology, and it is in a certain sense finer than the Zariski one. In particular, as we will see, Grothendieck topologies allow to generalize the notion of sheaf.

The phenomenon that consists in constructing global objects by gluing local data that satisfy suitable cocycle conditions is called descent. Descent theory is fundamental in Algebraic Geometry. In order to recall it, we will first introduce the formalism of fibered categories.

Afterwards, we define stacks, which are roughly speaking those fibered categories in which descent theory works. Of greater interest are algebraic stacks. Briefly, are those stacks $\mathcal{M}$ which own a smooth atlas $U \to \mathcal{M}$, where U is a scheme, such that for each other morphism $Y \to \mathcal{M}$ with Y scheme, the pullback $U \times_{\mathcal{M}} Y$ is a scheme too. This implies that $\mathcal{M}$ inherits the geometric properties from the schemes.

We will recall in detail the quotient stacks, which are of crucial importance both because the Deligne-Mumford stacks are locally described as quotients, and because the stack of elliptic curve is globally a quotient stack.

Finally, since the objective of the thesis is to give a presentation of the computation of the Picard group of the stack of elliptic curves, we will conclude the chapter recalling how the notion of quasi-coherent sheaf can be extended to stacks and, above all, to quotient stacks.

This chapter is almost completely taken from the works of A. Vistoli [Vis04] and M. Olsson [Ols16].

1.1. Some properties of morphisms of schemes

In this section we recall some fundamental properties of morphisms of schemes: smooth, unramified and étale morphisms. They are crucial in Algebraic Geometry for several reasons. One of the main reasons is that, as Proposition 1.1.5 suggests, they allow to global extend some local properties of morphisms of schemes. They form the basis of the study of the arguments of descent we will see in the next sections.

Definition 1.1.1. A morphism of schemes $f: X \to Y$ is *locally of finite presentation* if for any $x \in X$ there are affine neighbourhoods U of x in X and V of $f(x)$ in Y such that $f(U) \subset V$ and $\mathcal{O}(U)$ is finitely presented (as an algebra) over $\mathcal{O}(V)$.

This property of morphisms of schemes is a kind of generalization of being locally of finite type. Actually, when Y is locally noetherian, then f is locally of finite presentation if and only if it is locally of finite type.

Definition 1.1.2. Let $f: X \to Y$ be a morphism of schemes. We say that f is *formally smooth* (respectively *formally unramified, formally étale*) if for every affine Y-scheme $Y' \to Y$ and every closed embedding $Y'_0 \to Y'$ defined by a nilpotent ideal, the map

$$Hom_Y(Y', X) \to Hom_Y(Y_0, X)$$

is surjective (respectively injective, bijective).

Observation 1.1.3. To visualize better the definition, observe the diagram

$$\begin{tikzcd} Y'_0 \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] \arrow[ur, dashed, "h"] & Y. \end{tikzcd}$$

The definition states that f is formally smooth (respectively formally unramified, formally étale) if for every g making the diagram above commute, there exists at least (respectively at most, exactly) a function h such that the diagram commutes.

Definition 1.1.4. If f is locally of finite presentation and formally smooth (respectively formally unramified, formally étale), then f is called *smooth* (respectively *unramified, étale*).

Although the three types of morphisms (unramified, smooth, and étale) can be defined directly through properties which could suggest better the geometric intuition behind these, in practice the lifting properties are very useful, above all each time we need a functorial point of view.

Let $f: X \to Y$ be a morphism of schemes. The majority of reasonable properties of morphisms of schemes which f could satisfy, is stable under base change, but the "opposite" is not true in general, i. e. these properties are not stable for descent. But if we add some further powerful finiteness conditions to the morphism $Y' \to Y$ for which we want to pull back, this becomes true.

Proposition 1.1.5. Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a cartesian diagram of schemes in which $Y' \to Y$ is faithfully flat and either quasi-compact or locally of finite presentation. Consider one of the following properties:

- is separated,
- is quasi-compact,
- is locally of finite-presentation,
- is proper,
- is affine,
- is finite,
- is flat,
- is smooth,

- is unramified,
- is étale,
- is an embedding,
- is a closed embedding.

Then $X \to Y$ has of these properties if and only if $X' \to Y'$ has.

Proof. See [Vis04], Proposition 1.15. □

In the thesis we will not use the descent for all these properties, but they have been mentioned anyway to emphasize the importance in the development of the theory of faithfully flat and either quasi-compact or locally of finite presentation morphisms.

1.2. Sheaves in Grothendieck topologies

Let X be a topological space, and let Op_X denote the category in which the objects are the open subsets of X and the morphisms are the inclusions. Then a presheaf of sets on X is a functor $(Op_X)^{op} \to (Set)$. We know that this functor is a sheaf when it satisfies the gluing conditions.

Grothendieck noticed that the definition of sheaf works in practice because there are the intersections (that is the fiber product) in the category Op_X and therefore could be adapted to a generic category. Before generalizing the notion of sheaf, he provided to integrate the notion of topology. The elegant aspect of its method derives from the fact that he did not redefine the notion of opens subset, but the notion of covering.

Definition 1.2.1. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is the following data: given an object U of C, we assign to U collections of arrows $\{U_i \to U\}$. They are called *coverings* of U. The following axioms must be satisfied.

- If $V \to U$ is an isomorphism, then the set $\{V \to U\}$ is a covering.
- If $\{U_i \to U\}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then the fibered products $\{U_i \times_U V\}$ exist, and the family of projections $\{U_i \times_U V \to V\}$ is a covering.
- If $\{U_i \to U\}$ is a covering, and for each index i we have a covering $\{V_{ij} \to U_i\}$, the collection of composites $\{V_{ij} \to U_i \to U\}$ is a covering of U.

A category with a Grothendieck topology is called *site*.

Example 1.2.2. Let X be a topological space; let Op_X be the category defined above. Then we get a Grothendieck topology on Op_X by associating with each open subset U the set of open coverings of U.

We see now some examples of topologies that one can put on the category $(Sch(S))$ of schemes over S.

Example 1.2.3 (The global Zariski topology)**.** Given a scheme U, a covering $\{U_i \to U\}$ for U is a family of open embeddings that covers U. Here we must interpret as meaning that a morphism $U_i \to U$ gives an isomorphism of U_i with an open subscheme V_i of U, and the V_i's cover U in the usually sense.

Example 1.2.4 (The global étale topology)**.** A covering $\{U_i \to U\}$ is a jointly surjective collection of étale maps.

Example 1.2.5 (The fppf topology)**.** A covering $\{U_i \to U\}$ is a jointly surjective collection of flat maps locally of finite presentation. The abbreviation fppf stands for "fidèlement plat et de présentatione finie".

Sometimes one may be interested in considering coverings that are not locally finitely presented. At the same time, we would like that the coverings behave well, hence we need some finiteness conditions. Proposition 1.1.5 seems to suggest it might be fine assume the maps being faithfully flat and quasi-compact. But then, Zariski covers would not be included, and the resulting topology would not be comparable with the Zariski topology. Hence, the definition which follows gives the correct sheaf theory.

Proposition 1.2.6. Let $f: X \to Y$ be a surjective morphism of schemes. Then the following properties are equivalent:

- Every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.
- There exists a covering $\{V_i\}$ of Y by open affine subschemes, such that each V_i is the image of a quasi-compact open subset of X.
- Given a point $x \in X$, there exists an open neighbourhood U of x in X, such that the image $f(U)$ is open in Y, and the restriction $U \to f(U)$ of f is quasi-compact.
- Given a point $x \in X$, there exists a quasi-compact open neighbourhood U of x in X, such that the image $f(U)$ is open and affine in Y.

Proof. See [Vis04], Proposition 2.33. □

Definition 1.2.7. An *fpqc morphism of schemes* is a faithfully flat morphism that satisfies the equivalent conditions of Proposition 1.2.6

The abbreviation fpqc stands for "fidèlement plat et quasi-compact".
We list here some basic properties of fpqc morphisms.

Proposition 1.2.8.
- The composition of two fpqc morphisms is fpqc.
- If $f: X \to Y$ is a morphism of schemes, and there is an open covering V_i of Y, such that the restriction $f^{-1}(V_i) \to V_i$ is fpqc, then f is fpqc.
- An open faithfully flat morphism is fpqc.
- A fppf morphism is fpqc.
- Being fpqc is a property stable under base change.
- If $f: X \to Y$ is an fpqc morphism, a subset of Y is open if and only if $f^{-1}(U)$ is open in X.

Proof. See [Vis04], Proposition 2.35. □

Example 1.2.9 (The fpqc topology)**.** Let U be a S-scheme. Then a covering of U is a family of morphisms $\{U_i \to U\}$ such that the induced morphism $\bigsqcup U_i \to U$ is fpqc.

Remark 1.2.10. The fpqc topology is finer than the fppf topology, which is finer than the étale topology, which is finer than the Zariski topology.

Fortunately for fpqc morphisms, we do not lose the appreciable properties of Proposition 1.1.5, i.e. many properties are local on the codomain in the fpqc topology.

Proposition 1.2.11. Let $X \to Y$ be a morphism of schemes, $\{Y_i \to Y\}$ be an fpqc covering. Consider the following properties of morphisms of scheme:

- is separated,
- is quasi-compact,
- is locally of finite-presentation,
- is proper,
- is affine,
- is finite,
- is flat,
- is smooth,

- is unramified,
- is étale,
- is an embedding,
- is a closed embedding.

Then $X \to Y$ has one of these properties if and only if for all $i \in I$, $Y_i \times_Y X \to Y_i$ has it too.

Proof. See [Vis04], Proposition 2.36. □

We present now the Grothendieck generalization of the concept of sheaf.

Definition 1.2.12. Let $\mathcal{C}$ be a site, $F \colon \mathcal{C}^{op} \to (Set)$ a presheaf.

- F is said to be *separated* if, given a covering $\{U_i \to U\}_{i \in I}$, the map $F(U) \to \prod_{i \in I} F(U_i)$ is injective.
- F is a *sheaf* if for every object $U \in \mathcal{C}$ and covering $\{U_i \to U\}_{i \in I}$ the sequence

$$F(U) \xrightarrow{f} \prod_{i \in I} F(U_i) \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} \prod_{i,j \in I} F(U_i \times_U U_j)$$

  is exact, i. e. the map f identifies $F(U)$ with the equalizer of the two projections.

Remark 1.2.13. As in the classical case of topological spaces we can talk about sheaves of groups, rings, modules over a ring, etc. on a site.

Remark 1.2.14. The usual construction of sheafification of a presheaf of sets on a topological spaces carries over to this more general context. For the details, see Paragraph 2.3.7 of [Vis04].

In the end, we want to emphasize that sometimes two different topologies on the same category define the same sheaves.

Definition 1.2.15. Let $\mathcal{C}$ be a category, $\{U_i \to U\}_i$ a set of arrows. A *refinement* $\{V_\alpha \to U\}_\alpha$ is a set of arrows such that for each index α there is some i such that $V_\alpha \to U$ factors through $U_i \to U$.

Definition 1.2.16. Let $\mathcal{C}$ be a category, $\mathcal{T}$ and $\mathcal{T}'$ two topologies on $\mathcal{C}$. We say that $\mathcal{T}$ is *subordinate* to $\mathcal{T}'$ if every covering in $\mathcal{T}$ has a refinement that is a covering in $\mathcal{T}'$. In the case in which every covering of in $\mathcal{T}'$ has a refinement in $\mathcal{T}$ too, we say that $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent.*

Proposition 1.2.17. Two topologies $\mathcal{T}$ and $\mathcal{T}'$ on a category $\mathcal{C}$ are equivalent if and only if they have the same sheaves.

Proof. See [Vis04], Proposition 2.49. □

Example 1.2.18. The last proposition is the reason for which we have not defined the *smooth topology* over (Sch/S), in which coverings $\{U_i \to U\}$ are jointly surjective set of smooth morphisms.

In fact, by Corollary 17.16.3 in [Gro67], given a smooth covering $\{U_i \to U\}$ we can find an étale surjective morphism $f \colon V \to U$ that factors through the disjoint union $\coprod U_i \to U$. Then, $\{f^{-1}(U_i) \to U\}$ is an étale covering that is a refinement of the previous one.

Since obviously every étale covering is a smooth cover, the two topologies are equivalent.

1.3. Fibered categories

Fix a category $\mathcal{C}$. In this section we study a phenomenon that concerns categories over $\mathcal{C}$, that is, the data of a category $\mathcal{F}$ and of a functor $\rho_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{C}$.

Definition 1.3.1. Let $\mathcal{F}$ be a category over $\mathcal{C}$. An arrow $\Phi\colon \xi \to \nu$ of $\mathcal{F}$ is *cartesian* if for any arrow $\Psi\colon \chi \to \nu$ in $\mathcal{F}$ and any arrow $h\colon \rho_{\mathcal{F}}\chi \to \rho_{\mathcal{F}}\xi$ in $\mathcal{C}$ with $\rho_{\mathcal{F}}\Phi \circ h = \rho_{\mathcal{F}}\Psi$, there exists a unique arrow $\vartheta\colon \chi \to \xi$ with $\rho_{\mathcal{F}}\vartheta = h$ and $\Phi \circ \vartheta = \Psi$, as in the commutative diagram

$$\begin{tikzcd}
\chi \arrow[rrd, bend left, "\Psi"] \arrow[rd, dashed, "\vartheta"] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\Phi"] \arrow[dd, mapsto] & \nu \arrow[dd, mapsto] \\
\rho_{\mathcal{F}}\chi \arrow[rd, "h"] \arrow[rrd, bend left] & & \\
& \rho_{\mathcal{F}}\xi \arrow[r] & \rho_{\mathcal{F}}\nu .
\end{tikzcd}$$

If $\xi \to \nu$ is a cartesian arrow of $\mathcal{F}$ mapping to an arrow $U \to V$ of $\mathcal{C}$, we also say that ξ is *a pullback of ν to U*.

Remark 1.3.2. Given two pullback $\Phi\colon \xi \to \nu$ and $\tilde{\Phi}\colon \tilde{\xi} \to \nu$ of ν to U, the unique arrow $\vartheta\colon \tilde{\xi} \to \xi$ that fits into the diagram

$$\begin{tikzcd}
\tilde{\xi} \arrow[rrd, bend left, "\tilde{\Phi}"] \arrow[rd, dashed, "\vartheta"] \arrow[dd, mapsto] & & \\
& \xi \arrow[r, "\Phi"] \arrow[dd, mapsto] & \nu \arrow[dd, mapsto] \\
U \arrow[rd, equal] \arrow[rrd, bend left] & & \\
& U \arrow[r] & V .
\end{tikzcd}$$

is an isomorphism.

In other words, a pullback is unique, up to a unique isomorphism.

Definition 1.3.3. A *fibered category over* $\mathcal{C}$ is a category $\mathcal{F}$ over $\mathcal{C}$, such that for every arrow $f\colon U \to V$ in $\mathcal{C}$ and an object ν of $\mathcal{F}$ mapping to V, there is a cartesian arrow $\Phi\colon \xi \to \nu$ with $\rho_{\mathcal{F}}\Phi = f$.

Definition 1.3.4. If $\mathcal{F}$ and $\mathcal{G}$ are fibered categories over $\mathcal{C}$, then a *morphism of fibered categories* $F\colon \mathcal{F} \to \mathcal{G}$ is a functor such that:

- F is *base-preserving*, that is $\rho_{\mathcal{G}} \circ F = \rho_{\mathcal{F}}$.
- F sends cartesian arrows to cartesian arrows.

We use the following notation: given a fibered category $\mathcal{F}$ over $\mathcal{C}$ and given an object $U \in \mathcal{C}$ we denote with $\mathcal{F}(U)$ the subcategory of $\mathcal{F}$ whose objects are the objects ξ of $\mathcal{F}$ such that $\rho_{\mathcal{F}}\xi = U$, and whose morphisms are morphisms Φ in $\mathcal{F}$ with $\rho_{\mathcal{F}}\Phi = id_U$.

In this thesis we are fundamentally interested in categories fibered in groupoids.

Definition 1.3.5. A *category fibered in groupoids over* $\mathcal{C}$ is a category $\mathcal{F}$ fibered over $\mathcal{C}$, such that the category $F(U)$ is a groupoid for any object U of $\mathcal{C}$. A groupoid is a category in which every morphism is invertible.

Proposition 1.3.6. Let $\mathcal{F}$ be a category over $\mathcal{C}$. Then $\mathcal{F}$ is fibered in groupoids if and only if the following two conditions hold.

- Every morphism in $\mathcal{F}$ is cartesian.
- For each object ν of $\mathcal{F}$ and morphism $f\colon U \to \rho_{\mathcal{F}}\nu$ of $\mathcal{C}$, there exists a morphism $\Phi\colon \xi \to \nu$ of $\mathcal{F}$ with $\rho_{\mathcal{F}}\Phi = f$.

Proof. See [Vis04], Proposition 3.22. □

Corollary 1.3.7. Any base-preserving functor from a fibered category to a category fibered in groupoids is a morphism.

Proof. This is trivial, by the previous characterization. □

We are interested in defining the fiber product of categories fibered in groupoids.

First of all, we must consider fiber products of groupoids. Let

$$\begin{array}{ccc} & & \mathcal{G}_1 \\ & & \big\downarrow{\scriptstyle f} \\ \mathcal{G}_2 & \xrightarrow{g} & \mathcal{G} \end{array}$$

be a diagram of groupoids. The groupoid

$$\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \tag{1.1}$$

is defined as follows. The objects of $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ are triples (x, y, σ) where $x \in \mathcal{G}_1$ and $y \in \mathcal{G}_2$ are objects and

$$\sigma\colon f(x) \to g(y)$$

is an isomorphism in $\mathcal{G}$. A morphism

$$(x', y', \sigma') \to (x, y, \sigma)$$

is a pair of isomorphisms $a\colon x \to x'$ and $b\colon y \to y'$ such that the diagram

$$\begin{array}{ccc} f(x') & \xrightarrow{\sigma'} & g(y') \\ \big\downarrow{\scriptstyle f(a)} & & \big\downarrow{\scriptstyle g(b)} \\ f(x) & \xrightarrow{\sigma} & g(y) \end{array}$$

commutes.

There is a natural isomorphism of functors

$$\Sigma\colon f \circ pr_1 \to g \circ pr_2,$$

where pr_i denote the projections. The category $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ together with the functors pr_1 and pr_2, and the isomorphism Σ have the following universal property. Suppose $\mathcal{H}$ is another groupoid and that

$$\alpha\colon \mathcal{H} \to \mathcal{G}_1, \ \ \beta\colon \mathcal{H} \to \mathcal{G}_2 \ \ \gamma\colon f \circ \alpha \to g \circ \beta$$

are two functors and γ is an isomorphism of functors. Then there exists a collection of data

$$(h\colon \mathcal{H} \to \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2,\, \lambda_1,\, \lambda_2)$$

where h is a functor,

$$\lambda_1 : \alpha \to pr_1 \circ h, \ \ \lambda_2 \circ \beta \to pr_2 \circ h$$

are isomorphisms of functors, and the diagram

$$\begin{array}{ccc} f \circ \alpha & \xrightarrow{f(\lambda_1)} & f \circ pr_1 \circ h \\ \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \Sigma \circ h} \\ g \circ \beta & \xrightarrow{g(\lambda_2)} & g \circ pr_2 \circ h \end{array}$$

commutes. The data

$$(h, \lambda_1, \lambda_2)$$

is unique up to unique isomorphism.

We consider now the fiber product of categories fibered in groupoids. Let $\mathcal{C}$ be a category and let

$$\begin{array}{ccc} & & \mathcal{F}_1 \\ & & \big\downarrow{\scriptstyle c} \\ \mathcal{F}_2 & \xrightarrow{d} & \mathcal{F}_3 \end{array}$$

be a diagram of categories fibered in groupoids over $\mathcal{C}$.

Consider a category $\mathcal{G}$ fibered in groupoids over $\mathcal{C}$, morphism of fibered categories

$$\alpha\colon \mathcal{G} \to \mathcal{F}_1, \quad \beta\colon \mathcal{G} \to \mathcal{F}_2$$

and an isomorphism $\gamma\colon c\circ\alpha \to d\circ\beta$ of morphisms of fibered categories $\mathcal{G} \to \mathcal{F}_3$. We must clarify what we mean here for transformation of morphism of fibered categories. It is a natural transformation of functors such that for every $x \in \mathcal{G}$, the morphism $\gamma_x\colon c\circ\alpha(x) \to d\circ\beta(x)$ in $\mathcal{F}_3$ projects to the identity morphism in $\mathcal{C}$. In other words, γ_x is a morphism in $\mathcal{F}_3(\rho_{\mathcal{G}}(x))$. Giving the data (α, β, γ) is equivalent to giving an object of

$$Hom_{\mathcal{C}}(\mathcal{G}, \mathcal{F}_1) \times_{Hom_{\mathcal{C}}(\mathcal{G}, \mathcal{F}_3)} Hom_{\mathcal{C}}(\mathcal{G}, \mathcal{F}_2).$$

Such data defines for any other category fibered in groupoids $\mathcal{H}$ a morphism of groupoids

$$\begin{aligned} &Hom_{\mathcal{C}}(\mathcal{H}, \mathcal{G}) \to Hom_{\mathcal{C}}(\mathcal{G}, \mathcal{F}_1) \times_{Hom_{\mathcal{C}}(\mathcal{G}, \mathcal{F}_3)} Hom_{\mathcal{C}}(\mathcal{G}, \mathcal{F}_2) \\ &(h\colon \mathcal{H} \to \mathcal{G}) \mapsto (\alpha\circ h,\ \beta\circ h,\ \gamma\circ h). \end{aligned} \tag{1.2}$$

Proposition 1.3.8. ▶ There exists a collection of data $(\mathcal{G}, \alpha, \beta, \gamma)$ as above, such that for every category fibered in groupoids $\mathcal{H}$ over $\mathcal{C}$ the map 1.2 is an isomorphism.

▶ If $(\mathcal{G}', \alpha', \beta', \gamma')$ is another collection of data as in the previous point, then there exists a triple (F, u, v) where $F\colon \mathcal{G} \to \mathcal{G}'$ is an equivalence of fibered categories, $u\colon \alpha \to \alpha'\circ F$ and $v\circ\beta \to \beta'\circ F$ are isomorphisms of morphisms of fibered categories, and the following diagram commutes:

$$\begin{tikzcd} c\circ\alpha \arrow[r, "cou"] \arrow[d, "\gamma"] & c\circ\alpha'\circ F \arrow[d, "\gamma'"] \\ d\circ\beta \arrow[r, "dov"] & d\circ\beta'\circ F. \end{tikzcd}$$

Moreover, if (F', u', v') is a second such triple, then there exists a unique isomorphism $\sigma\colon F' \to F$ such that the diagrams

$$\begin{tikzcd} \alpha \arrow[r, "u'"] \arrow[dr, "u"] & \alpha'\circ F' \arrow[d, "\sigma"] \\ & \alpha'\circ F \end{tikzcd}$$

and

$$\begin{tikzcd} \beta \arrow[r, "v'"] \arrow[dr, "v"] & \beta'\circ F' \arrow[d, "\sigma"] \\ & \beta'\circ F \end{tikzcd}$$

commute.

Proof. See [Ols16], Proposition 3.4.13 □

Definition 1.3.9. We write $\mathcal{F}_1 \times_{\mathcal{F}_3} \mathcal{F}_2$ for the fibered category in the first point of the Proposition 1.3.8. The quadruple $(\mathcal{F}_1 \times_{\mathcal{F}_3} \mathcal{F}_2, \alpha, \beta, \gamma)$ is said *the fiber product of $\mathcal{F}_1$ and $\mathcal{F}_2$ over $\mathcal{F}_3$*. We always suppress the data of the morphisms from the notation, since it is implicit.

Observation 1.3.10. In the case when $\mathcal{C}$ is the punctual category, the fiber product construction coincides with that of 1.1.

1.4. Descent of objects of fibered category and stacks

Descent theory is the natural generalization to sites of the more familiar gluing properties that one encounters in the study of schemes.

Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a category fibered over $\mathcal{C}$. Given a covering $\{U_i \to U\}$, set $U_{ij} := U_i \times U_j$ and $U_{ijk} := U_i \times_U U_j \times_U U_k$ for each triple of indices i, j and k.

Definition 1.4.1. Let $\mathcal{U} = \{\sigma_i : U_i \to U\}$ be a covering in $\mathcal{C}$. We define the *category of descent data* associated to the cover $\mathcal{U}$, denoted by $\mathcal{F}(\mathcal{U}) = \mathcal{F}(\{U_i \to U\})$, in the following way. The objects are $(\{\xi_i\}, \{\Phi_{ij}\})$, where (as the indices vary) ξ_i is an object in $F(U_i)$, and Φ_{ij} is an isomorphism $\Phi_{ij} : pr_2^*\xi_j \cong pr_1^*\xi_i$ in $F(U_{ij})$. These morphisms must satisfy the following cocycle condition.

For any triple of index i, j and k, we have the equality

$$pr_{13}^*\Phi_{ik} = pr_{12}^*\Phi_{ij} \circ pr_{23}^*\Phi_{jk} : pr_3^*\xi_k \to pr_1^*\xi_i,$$

where the pr represent the corresponding projections. The isomorphisms Φ_{ij} are called *transition isomorphisms* of the object with descent data.

Given two objects $(\{\xi_i\}, \{\Phi_{ij}\})$ and $(\{\nu_i\}, \{\Psi_{ij}\})$ in $\mathcal{F}(\mathcal{U})$, an arrow

$$\{\alpha_i\} : (\{\xi_i\}, \{\Phi_{ij}\}) \to (\{\nu_i\}, \{\Psi_{ij}\})$$

is a collection of morphisms $\alpha_i : \xi_i \to \nu_i$ in $F(U_i)$, with the property that for each pair of indices i, j, the diagram

$$\begin{array}{ccc} pr_2^*\xi_j & \xrightarrow{pr_2^*\alpha_j} & pr_2^*\nu_j \\ \downarrow{\scriptstyle \Phi_{ij}} & & \downarrow{\scriptstyle \Psi_{ij}} \\ pr_1^*\xi_i & \xrightarrow{pr_1^*\alpha_i} & pr_1^*\nu_i \end{array}$$

commutes.

The composition of morphisms is made in the obvious way.

There is a functor $\mathcal{F}(U) \to \mathcal{F}(\{\sigma_i : U_i \to U\}_i)$. Namely, for each object $\xi \in \mathcal{F}(U)$ we can construct the object with descent data $(\{\sigma_i^*\xi\}, \{\Phi_{ij}\})$, where the isomorphisms $\Phi_{ij} : pr_2^*\sigma_j^*\xi \cong pr_1^*\sigma_i^*\xi$ are those which come from the fact that both $pr_2^*\sigma_j^*\xi$ and $pr_1^*\sigma_i^*\xi$ are pullback of ξ to U_{ij}. Given an arrow $\alpha : \xi \to \nu$ in $\mathcal{F}(U)$, we get the natural arrows $\sigma_i^*\alpha$.

Remark 1.4.2. The category of descent data does not depend on the choice of fibered products U_{ij} and U_{ijk}, in the sense that with different choices we get isomorphic categories.

By the axiom of choice, every fibered category $\mathcal{F} \to \mathcal{C}$ has a *cleavage*, that is a class K of cartesian arrows in $\mathcal{F}$ such that for each arrow $f : U \to V$ in $\mathcal{C}$ and each object ν in $\mathcal{F}(V)$ there exists a unique arrow in K with target ν mapping to f in $\mathcal{C}$.

It is important to notice that the morphism $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ does not depend on the choice of a cleavage.

In this thesis we will often say "the pullback" instead of "a pullback", omitting the choice behind this. This is not very formal, it is possible to make the arguments more rigorous, as it is done for example in [Vis04], Paragraph 4.1.2.

We are finally ready to define the stacks.

Definition 1.4.3. Let $\mathcal{C}$ be a site. A category fibered in groupoids $\rho : \mathcal{F} \to \mathcal{C}$ is a *stack* if for every object $U \in \mathcal{C}$ and covering $\{U_i \to U\}$, the functor

$$\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\}) \tag{1.3}$$

is an equivalence of categories.

If the functor is fully faithful, the category is said to be a *prestack*.

We give now some definitions to give an alternative formulation of the previous concept.

Definition 1.4.4. An object with descent data $(\{\xi_i\}, \{\Phi_{ij}\})$ in $\mathcal{F}(\{U_i \to U\})$ is *effective* if it is isomorphic to the image of an object of $\mathcal{F}(U)$.

Hence a prestack is a stack if and only if every object with descent data is effective.

Let $U \in \mathcal{C}$ be an object and let $(\mathcal{C}/U)$ the comma category, i. e. the objects are the morphism $V \to U$ and the arrows are the commutative diagram as the following:

$$\begin{array}{ccc} V_1 & \longrightarrow & V_2 \\ & \searrow \quad \swarrow & \\ & U. & \end{array}$$

Consider the comma topology, in which the covers of $V \to U$ are of the type $\{V_i \to U\}_i$, with $\{V_i \to V\}_i$ a cover of $\mathcal{C}$ and making the diagram

$$\begin{tikzcd} V_i \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & U & \end{tikzcd}$$

commute for all i.

Let $\xi, \nu \in \mathcal{F}(U)$ be two objects in the fiber over U. We can define a presheaf

$$\underline{Isom}(\xi, \nu) \colon (\mathcal{C}/U)^{op} \to (Set)$$

as follows.

For any morphism $f \colon V \to U$, choose pullbacks $f^*\xi$ and $f^*\nu$, and set

$$\underline{Isom}(\xi, \nu)(f \colon V \to U) := Isom_{\mathcal{F}(V)}(f^*\xi, f^*\nu).$$

For a composition

$$Z \xrightarrow{g} V \xrightarrow{f} U,$$

the pullback $(fg)^*\xi$ (respectively $(fg)^*\nu$), is a pullback along g of $f^*\xi$ (respectively $f^*\nu$), and therefore there is a canonical map

$$g^* \colon \underline{Isom}(\xi, \nu)(f \colon V \to U) \to \underline{Isom}(\xi, \nu)(fg \colon Z \to U)$$

compatible with composition.

Proposition 1.4.5. Let $\mathcal{F}$ be a category fibered in groupoid over a site $\mathcal{C}$. Then $\mathcal{F}$ is a prestack if and only if for any object U of $\mathcal{C}$ and any two object ξ and ν in $\mathcal{F}(U)$, the functor $\underline{Isom}(\xi, \nu) \colon (C/U)^{op} \to (Set)$ is a sheaf in the comma topology.

Proof. See [Vis04], Proposition 4.7. □

Remark 1.4.6. Recall that the fibered product of stacks is simply the fibered product as categories fibered in groupoids. Actually, let

$$\begin{tikzcd} & \mathcal{F}_1 \arrow[d] \\ \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \end{tikzcd}$$

be a diagram of stacks fibered in groupoids over $\mathcal{C}$. Then the fiber product $\mathcal{F}_1 \times_{\mathcal{F}_3} \mathcal{F}_2$ is a stack too. This follows from noting that for any covering $\{U_i \to U\}$ in $\mathcal{C}$ the maps

$$(\mathcal{F}_1 \times_{\mathcal{F}_3} \mathcal{F}_2)(U) \to \mathcal{F}_1(U) \times_{\mathcal{F}_3(U)} \mathcal{F}_2(U)$$

and

$$(\mathcal{F}_1 \times_{\mathcal{F}_3} \mathcal{F}_2)(\{U_i \to U\}) \to \mathcal{F}_1(\{U_i \to U\}) \times_{\mathcal{F}_3(\{U_i \to U\})} \mathcal{F}_2(\{U_i \to U\})$$

are equivalences of groupoids.

At this point, we give an example of important result of descent theory, in particular for what concerns quasi-coherent sheaves.

Theorem 1.4.7. Let S be a scheme. The fibered category $(QCoh/S)$ over(Sch/S), whose fiber of a scheme U over S is the category $QCoh(U)$ of quasi-coherent sheaves on U, is a stack with respect to the fpqc topology.

Proof. For a proof, see [Vis04] Theorem 4.23. □

This is quite considerable. Actually, it is a standard fact of the classic Algebraic Geometry that quasi-coherent sheaves are sheaves in the Zariski topology. But Zariski topology is much coarser then fpqc, so a priori one would not expect this to happen. This result is very powerful and it carries to many generalizations, such as Theorem 2.2.10 in Chapter 2, which for us is essential to prove our results on the stack of elliptic curves.

1.5. Torsors

Definition 1.5.1. Let C be a site and let μ be a sheaf of groups on C. We say that a sheaf $\mathcal{P}$ on C is a *μ-torsor* if there is a left action ρ of μ on $\mathcal{P}$, such that the following conditions hold:

- For every $U \in C$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{P}(U_i) \neq \emptyset$ for all i.
- The map

$$\begin{aligned} \mu \times \mathcal{P} &\to \mathcal{P} \times \mathcal{P}, \\ (g, p) &\mapsto (p, gp) \end{aligned} \tag{1.4}$$

  is an isomorphism.

Observe that in light of the second condition, when $\mathcal{P}(U)$ is nonempty the action of $\mu(U)$ on $\mathcal{P}(U)$ is simply transitive. We say that a torsor $(\mathcal{P}, \rho)$ is a *trivial torsor* if $\mathcal{P}$ has a global section p, which implies in particular that we have an isomorphism

$$\begin{aligned} \mu &\to \mathcal{P} \\ g &\mapsto gp. \end{aligned}$$

A morphism of μ-torsors $(\mathcal{P}, \rho) \to (\mathcal{P}', \rho')$ is simply a μ-equivariant morphism of sheaves $f\colon \mathcal{P} \to \mathcal{P}'$.

The notion of torsor is closely related to the more classical notion of principal bundle. To explain this, fix a scheme X and consider the site (Sch/X) of schemes over X with the fppf topology. This is the common choice of topology, since in the Zariski topology there are few open sets. Assume μ is representable by a flat locally finitely presented affine X-group scheme G.

Definition 1.5.2. A *principal G-bundle* over X is a pair $(\pi\colon P \to X, \rho)$, where π is flat, locally finitely presented, surjective morphism of schemes, and $\rho\colon G \times_X P \to P$ is a morphism such that the following axioms hold:

- The diagram

$$\begin{tikzcd} G \times_X G \times_X P \arrow[r, "id_G \times \rho"] \arrow[d, "m \times id_P"] & G \times_X P \arrow[d, "\rho"] \\ G \times_X P \arrow[r, "\rho"] & P \end{tikzcd} \tag{1.5}$$

  commutes, where m is the map defining the group law of G.

- If $e\colon X \to G$ is the identity section, then the diagram

$$\begin{tikzcd} P \arrow[r, "(e\circ\pi, id_P)"] \arrow[rr, bend right, "id_P"'] & G \times_X P \arrow[r, "\rho"] & P \end{tikzcd}$$

  commutes.

- The map

$$(\rho, pr_2)\colon G \times_X P \to P \times_X P \tag{1.6}$$

  is an isomorphism.

A morphism of principal G-bundles $(P, \rho) \to (P', \rho')$ is a morphism of X-schemes $f\colon P \to P'$ such that the diagram

$$\begin{tikzcd} G \times_X P \arrow[r, "id_G \times f"] \arrow[d, "\rho"] & G \times_X P' \arrow[d, "\rho'"] \\ P \arrow[r, "f"] & P' \end{tikzcd}$$

commutes.

Observation 1.5.3. Let (P, ρ) be a principal G-bundle. We obtain a μ-torsor $(\mathcal{P}, \rho)$, by letting $\mathcal{P}$ be the sheaf represented by P. The action is that induced by ρ. Observe that in this way by the first two conditions in the Definition 1.5.2, follows that $\rho\colon \mu \times \mathcal{P} \to \mathcal{P}$ is a left action. The third condition in 1.5.2 implies the second in 1.5.1. Finally the first condition in the definition of Torsors (1.5.1) follows from the fact that locally in the fppf topology P has sections, i.e. there exists an fppf cover $\{X_i \to X\}$ with $\mathcal{P}(X_i) \neq \emptyset$ for all i. Moreover the morphisms between torsors and between G-bundles are both equivariant by definition. Hence this defines a fully faithful functor

$$(principal \;\; G - bundles) \to (\mu - torsors \;\; on \;\; X). \tag{1.7}$$

. Under certain hypothesis this is an equivalence, as we can see in the following proposition.

Proposition 1.5.4. If the structure morphism $G \to X$ is affine, then 1.7 is an equivalence of categories.

Proof. See [Ols16], Proposition 4.5.6. □

1.6. Algebraic Stacks

In this section we want to define algebraic stacks.

In order to define algebraic stacks, we need to present another fundamental preliminary notion, that is the notion of algebraic space. By definition, an algebraic space over a scheme S is a sheaf on the global étale site of the category of S-schemes, which satisfies certain properties similar to the properties needed to define the notion of scheme from the notion of affine scheme. One way to think about algebraic spaces is using étale equivalence relations. If X is a scheme over a base S, an equivalence relation on X is a monomorphism $R \hookrightarrow X \times_S X$ such that for every S-scheme T the T-valued points $R(T) \subset X(T) \times X(T)$ is an equivalence relation. We want to consider just étale equivalence relations, that are those equivalence relations R for which the two projections $R \to X$ are étale. For such an equivalence relation, one can form the sheaf X/R on the big étale site by sheafifying the presheaf sending T to the quotient of $X(T)$ by the equivalence relation $R(T)$. The sheaf X/R is an algebraic space, and every algebraic space can be described in this way. Exploiting this definition is easy to produces a lot of examples of algebraic spaces. By the way, this definition is often difficult to work with in practice. Fortunately, there is the possibility to give (as we do below) a more global definition of algebraic space, which is equivalent to the previous one.

Definition 1.6.1. Let S be a scheme and let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on (Sch/S) with the étale topology.

- f is *representable by schemes* if for every S-scheme T and morphism $T \to \mathcal{G}$ the fiber product $\mathcal{F} \times_{\mathcal{G}} T$ is a scheme.
- A property P of morphism of schemes is said to be *stable* if for all $f\colon X \to Y$ in the site $\mathcal{C}$ and covering $\{Y_i \to Y\}$, the morphism f satisfies P if and only if all the maps $X \times_Y Y_i \to Y_i$ satisfy P.
- Let P be a stable property of morphisms of schemes. If f is representable by schemes, we say that f *has property* P if for every S-scheme T the morphism of schemes $pr_2\colon \mathcal{F} \times_{\mathcal{G}} T \to T$ has property P.

Remark 1.6.2. In the above definition, and we will do it for the entire thesis, we will abusively write T both for the scheme and the corresponding sheaf $Hom_{(Sch/S)}(\bullet, T)$.

Lemma 1.6.3. Let S be a scheme and let $\mathcal{F}$ be a sheaf on (Sch/S) with the étale topology. Suppose that the diagonal morphism $\Delta\colon \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable by schemes. Then if T is a scheme, any morphism $f\colon T \to \mathcal{F}$ is representable by schemes.

Proof. Indeed, if T and T' are schemes, the fiber product of the diagram

$$\begin{tikzcd} & T \arrow[d, "f"] \\ T' \arrow[r, "g"] & \mathcal{F} \end{tikzcd}$$

is isomorphic to the fiber product of the diagram

$$\begin{tikzcd} & T \times_S T' \arrow[d, "f \times g"] \\ \mathcal{F} \arrow[r, "\Delta"] & \mathcal{F} \times \mathcal{F}, \end{tikzcd}$$

and since Δ is representable by schemes the fiber product is a scheme. □

Definition 1.6.4. Let S be a scheme. We say that a functor $X\colon (Sch/S)^{op} \to (Set)$ is an *algebraic space* over S, if the following hold:

- X is a sheaf in the big étale topology.
- The diagonal morphism $\Delta\colon X \to X \times_S X$ is representable by schemes.
- There exists an S-scheme $U \to S$ and a surjective étale morphism $U \to X$.

Notice that the third condition makes sense in light of the second one and of Lemma 1.6.3.

Example 1.6.5. Schemes over S are algebraic spaces over S.

In order to see some non trivial examples of algebraic spaces, the proof of the equivalence between this global definition and the definition provided giving an étale equivalence relation as mentioned above, and the description of the main properties of these spaces, we suggest the reader to read Chapter 5 in [Ols16]. The reason for which we do not develop the exposition about the algebraic spaces is that in this thesis we only need them to define in a way as complete and general as possible the algebraic stacks. However, even if several algebraic spaces appear in the definition of algebraic stack, the ones we will encounter in the thesis will all be defined through schemes (this is fortunate because they will be easier to study).

In this section, by *stack* we mean a stack in the sense of Definition 1.4.3 over the category of S-schemes with the fpqc topology.

Definition 1.6.6. A morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is *representable* if for every scheme U and morphism $y\colon U \to \mathcal{Y}$ the fiber product $\mathcal{X} \times_{\mathcal{Y},y} U$ is an algebraic space.

Definition 1.6.7. A stack $\mathcal{X}/S$ is an *algebraic stack* if the following hold:

- The diagonal
$$\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$$
is representable.
- There exists a smooth morphism $\pi\colon X \to \mathcal{X}$ with X a scheme.

A morphism of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of stacks.

Note that the diagonal being representable implies that every morphism $t\colon T \to \mathcal{X}$, with T a scheme, is representable, basically for the same argumentation made to prove Lemma 1.6.3. It therefore makes sense to talk about a smooth surjective morphism $X \to \mathcal{X}$.

Lemma 1.6.8. Let $\mathcal{X}/S$ be a stack over S. The diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable if and only if for every S-scheme U and two object $u_1, u_2 \in \mathcal{X}(U)$ the sheaf $\underline{Isom}(u_1, u_2)$ on $(Sch/S))$ is an algebraic space.

Proof. This is immediate from the definition and noting that we have a cartesian square

$$
\begin{tikzcd}
\underline{Isom}(u_1, u_2) \arrow[r] \arrow[d] & U \arrow[d, "u_1 \times u_2"] \\
\mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_S \mathcal{X}.
\end{tikzcd}
\tag{1.8}
$$

□

Proposition 1.6.9. Let $\mathcal{X}/S$ be an algebraic stack. Then for any diagram

$$
\begin{tikzcd}
 & X \arrow[d, "x"] \\
Y \arrow[r, "y"] & \mathcal{X},
\end{tikzcd}
$$

with X and Y algebraic spaces, the fiber product $X \times_{\mathcal{X}} Y$ is an algebraic space. In particular, any morphism $x\colon X \to \mathcal{X}$ from an algebraic space X to $\mathcal{X}$ is representable.

Proof. See [Ols16], Proposition 8.1.10. □

We resume now some basics facts about properties of algebraic stacks and morphisms between them.

Definition 1.6.10. Let P be a property of S-schemes, which is stable in the smooth topology. We say that an algebraic stack $\mathcal{X}/S$ has property P if there exists a smooth surjective morphism $\pi\colon X \to \mathcal{X}$ with X a scheme having property P.

For example, being locally noetherian, regular, locally of finite type over S, locally of finite presentation over S, are such P. Since we would like this to be an intrinsic property of the stack, it is natural to hope that it does not depend on the choice of the scheme. The following lemma assures us of this.

Lemma 1.6.11. Let P be a property of schemes which is stable with respect to the smooth topology, and let $\mathcal{X}/S$ be an algebraic stack having property P. Then for any smooth morphism $y\colon Y \to \mathcal{X}$ from an algebraic space Y, the space Y has property P.

Proof. See [Ols16], Lemma 8.2.4. □

We want to follow similar principal to define properties of morphisms of algebraic stacks. To do this, we introduce the following terminology. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over S. A *chart for f* is a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "g"] \arrow[rr, bend left, "h"] \arrow[dr, "q"'] & \mathcal{X}' \arrow[r, "f'"] \arrow[d, "p'"] & Y \arrow[d, "p"] \\
 & \mathcal{X} \arrow[r, "f"] & \mathcal{Y},
\end{tikzcd}
\tag{1.9}
$$

where X and Y are algebraic spaces, the square is cartesian, and g and p are smooth and surjective. In the case X and Y are schemes, then we call this *a chart for f by schemes*.

Definition 1.6.12. Let P be a property of morphisms of schemes which is stable and local on domain with respect to the smooth topology. We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ has property P if there exists a chart f by schemes such that the morphism h (1.9) has property P.

For example, P could be the property of being smooth, locally of finite presentation, surjective. It is natural to hope that this not depends on the choice of the chart. This is indeed true, as we can see in the following proposition.

Proposition 1.6.13. Let P be a property of morphisms of schemes which is stable and local on domain with respect to the smooth topology. Then a morphism of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$ has property P if and only if for every chart for f (1.9), the morphism h has property P.

Proof. See [Ols16], Proposition 8.2.8. □

Definition 1.6.14. Let P be a property of morphisms of algebraic spaces which is stable with respect to the smooth topology on the category of algebraic spaces over S. We say that a representable morphism of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$ has property P if for every morphism $Y \to \mathcal{Y}$ with Y an algebraic space, the morphism of algebraic spaces (since f is representable)

$$\mathcal{X} \times_{\mathcal{Y}} Y \to Y$$

has property P.

Observe that this definition is a particular case of definition 1.9, in which simply $\mathcal{X} \times_{\mathcal{Y}} Y$ is already an algebraic space.

For example, we can talk about a representable morphism of algebraic stacks being étale, smooth of relative dimension d, separated, proper, affine, finite, unramified, a closed/open embedding.

In particular if $f\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks over S, then the diagonal morphism

$$\Delta_{\mathcal{X}/\mathcal{Y}}\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \tag{1.10}$$

is representable, and we can make the following definition.

Definition 1.6.15. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over S, and let $\Delta_{\mathcal{X}/\mathcal{Y}}$ be the diagonal morphism. We say that:

- f is *quasi-separated* if the diagonal $\Delta_{\mathcal{X}/\mathcal{Y}}$ is quasi-compact and quasi-separated.
- f is *separated* if the diagonal $\Delta_{\mathcal{X}/\mathcal{Y}}$ is proper.

If $\mathcal{Y} = S$ and f is the structure morphism, then we simply say that $\mathcal{X}$ is *quasi-separated* (resp. *separated*).

We know present a fundamental type of stack.

Definition 1.6.16 (Deligne-Mumford stack)**.** An algebraic stack $\mathcal{X}/S$ is called *Deligne-Mumford* if there exists an étale surjection $X \to \mathcal{X}$ with X a scheme.

These stacks are the most studied and a lot of natural stacks are of this type. For example we will see in Chapter 2 that the stack of elliptic curves $\mathcal{M}_{1,1}$ is Deligne-Mumford.

The property of being formally unramified is stable and local on a domain with respect to the étale topology on the category of schemes, and stable with respect to the smooth topology. It therefore makes sens to talk about representable morphism of stacks being formally unramified.

Theorem 1.6.17. Let $\mathcal{X}/S$ be an algebraic stack. Then $\mathcal{X}$ is Deligne-Mumford if and only if the diagonal

$$\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$$

is formally unramified.

Proof. For a proof see [Ols16], Theorem 8.3.3, or the book of Laumon and Moret-Bailly [LMB18], proof of 8.1. □

We are not interested in going further into the topic of the Deligne-Mumford stacks, because this goes beyond the objectives of the thesis. For a more precise treatment see [Ols16] on Chapter 8.3. We will just use the fact that the result of the previous theorem can be interpreted as saying that a stack $\mathcal{X}$ is Deligne-Mumford if and only if the objects of $\mathcal{X}$ admit no infinitesimal automorphism. Precisely, let $\mathcal{X}/S$ be an algebraic stack and assume that the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is of finite presentation. Then the diagonal is formally unramified if and only if for every algebraically closed field k and object $x \in \mathcal{X}(k)$, the automorphism group scheme $\underline{Aut}_x$ is a reduced finite k-group scheme (i. e. a finite group). See [Ols16], Remark 8.3.4 for more details.

1.7. The algebraic stack $[X/G]$

We introduce now a fundamental example which, due to its importance, deserves a thorough discussion.

Definition 1.7.1. Let X be a S-scheme and G/S be a smooth group scheme which acts on X. Define $[X/G]$ to be the stack whose objects are triples $(T, \mathcal{P}, \pi)$ where

1. T is an S-scheme.
2. $\mathcal{P}$ is a $G_T := G \times_S T$-torsor on the big étale site of T.
3. $\pi\colon \mathcal{P} \to X \times_S T$ is a G_T-equivariant morphism of sheaves on (Sch/S).

A morphism $(T', \mathcal{P}', \pi') \to (T, \mathcal{P}, \pi)$ is a pair (f, f^b), where $f\colon T' \to T$ is an S-morphism of schemes and $f^b\colon \mathcal{P}' \to f^*\mathcal{P}$ is an isomorphism of $G_{T'}$-torsors on (Sch/T') such that the induced diagram

$$\begin{tikzcd} \mathcal{P}' \arrow[rr, "f^b"] \arrow[dr, "\pi'"'] & & f^*\mathcal{P} \arrow[dl, "f^*\pi"] \\ & X \times_S T' & \end{tikzcd}$$

commutes.

We now make a brief digression by presenting a definition and a preliminary lemma. Thanks to these, at the end of the section we see that $[X/G]$ is an algebraic stack and we prove a powerful criterion for establishing when a stack is indeed a quotient stack.

Definition 1.7.2. Let $X \to S$ be a stack over (Sch/S) (with the étale topology), and let $x\colon X \to \mathcal{X}$ be a morphism of stacks. Suppose the group G acts on X via $\chi\colon G \times_S X \to X$. We say that x is *G-invariant* if there exists a natural isomorphism $\alpha\colon x \circ \chi \to x \circ pr_2$ in the diagram

$$\begin{tikzcd} G \times_S X \arrow[r, "\chi"] \arrow[d, "pr_2"] & X \arrow[d, "x"] \\ X \arrow[r, "x"] & \mathcal{X}. \end{tikzcd}$$

Moreover, α must satisfy the condition that the diagram

$$\begin{tikzcd} x(\chi(g_1g_2, z)) \arrow[r, "\alpha_{(g_1g_2,z)}"] \arrow[d, "\alpha_{(g_1,\chi(g_2,z))}"'] & x(z) \arrow[d, equal] \\ x(\chi(g_2, z)) \arrow[r, "\alpha_{(g_2,z)}"] & x(z) \end{tikzcd}$$

commutes for all $g_1, g_2 \in G$ and $z \in X$.

Observation 1.7.3. Let $(X, pr_2\colon G \times_S X \to X, (\chi, pr_2))$ be the tautological torsor ($\chi\colon G \times_S X \to X$ is simply the action).

We can observe that the morphism $x\colon X \to [X/G]$ induced by the tautological torsor is G-invariant by construction.

Lemma 1.7.4. Let $t\colon T \to \mathcal{X}$ be a map where T denotes a scheme and $\mathcal{X}$ an algebraic stack. Let X be a scheme with an action χ of the group scheme G. The fiber product $X \times_{\mathcal{X}} T$ is an étale sheaf over T. If $x\colon X \to \mathcal{X}$ is G-invariant, then the action of G on X induces one of G_T on the sheaf $X \times_{\mathcal{X}} T$. Moreover:

- In the diagram

$$\begin{tikzcd}
X \times_{\mathcal{X}} T \arrow[dr, dotted, "\pi"] \arrow[drr, bend left, "pr_2"] \arrow[ddr, bend right, "pr_1"'] & & \\
& X \times_S T \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & T \arrow[d, "t"] \\
& X \arrow[r, "x"] & \mathcal{X} \arrow[dr] & \\
& & & S
\end{tikzcd}$$

pr_2 is G_T-invariant and π is G_T-equivariant.

- for all $v \colon V \to \mathcal{X}$, given an $\mathcal{X}$-map $f \colon T \to V$, the induced dashed morphism h

$$\begin{tikzcd}
X \times_{\mathcal{X}} T \arrow[dr, dotted, "h"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& T \times_V (X \times_{\mathcal{X}} V) \arrow[r] \arrow[d] & X \times_{\mathcal{X}} V \arrow[d] \\
& T \arrow[r, "f"] & V
\end{tikzcd}$$

is G_T-equivariant and makes the diagram

$$\begin{tikzcd}
X \times_{\mathcal{X}} T \arrow[rr, "h"] \arrow[dr, "\pi"'] & & T \times_V (X \times_{\mathcal{X}} V) \arrow[dl, "f^*\pi_V"] \\
& X \times_S T &
\end{tikzcd}$$

2-commutes, where π_V is the G_V-equivariant map defined in the first item.

Proof. As action on $X \times_{\mathcal{X}} T$, we define

$$\begin{aligned} \rho_T \colon G_T \times_T X \times_{\mathcal{X}} T &\to X \times_{\mathcal{X}} T \\ (g, z, y, \sigma) &\mapsto (\chi(g, z), y, \sigma \circ \alpha_{(g,z)}), \end{aligned}$$

where $z \in X$, $y \in T$ and $\sigma \colon x(z) \to t(y)$ is an isomorphism. Since α is associative, ρ_T is a G_T-action.

For the first point, the properties can be expressed as the commutativity of the squares

$$\begin{tikzcd}
(g, z, y, \sigma) \arrow[r, mapsto] \arrow[d, mapsto, "\rho_T"'] & (z, y, \sigma) \arrow[d, mapsto, "pr_2"] \\
(\chi(g, z), y, \sigma \circ \alpha_{(g,z)}) \arrow[r, mapsto, "pr_2"] & y
\end{tikzcd}$$

and

$$\begin{tikzcd}
(g, z, y, \sigma) \arrow[r, mapsto, "id_G \times \pi"] \arrow[d, mapsto, "\rho_T"'] & (g, z, y) \arrow[d, mapsto, "\chi \times id"] \\
(\chi(g, z), y, \sigma \circ \alpha_{(g,z)}) \arrow[r, mapsto, "\pi"] & (\chi(g, z), y),
\end{tikzcd}$$

which is actually true.

For the last item, h is the map which associates

$$(z, y, \sigma) \mapsto (y, (z, f(y), \beta_y^{-1} \circ \sigma)),$$

where β is the natural isomorphism of the commutative diagram

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "t"'] & & V \arrow[dl, "v"] \\ & \mathcal{X}. & \end{tikzcd}$$

Since the square

$$\begin{tikzcd} (g, z, y, \sigma) \arrow[r, "id_G \times h"] \arrow[d, "\rho_T"'] & (g, (y, (z, f(y), \beta_y^{-1} \circ \sigma))) \arrow[d, "f^*(\rho_V)"] \\ (\chi(g, z), y, \sigma \circ \alpha_{(g,z)}) \arrow[r, "h"] & (y, \chi(g, z), f(y), \beta_y^{-1} \circ \sigma \circ \alpha_{(g,z)})) \end{tikzcd}$$

commutes, h is G_T-invariant. Moreover, by the uniqueness of the map induced in the fiber product, we have $f^*(\pi_V) \circ h = \pi_T$. □

Theorem 1.7.5. $[X/G]$ is an algebraic stack.

Proof. Descent for sheaves implies that $[X/G]$ is a stack, see [Ols16], Proposition 4.2.12.

By [Sta22], Tag $[046K]$, if we find a smooth surjective atlas for $[X/G]$, then the diagonal $\Delta_{[X/G]}$ is representable.

To conclude it suffices to prove the following claim: the morphism $x \colon X \to [X/G]$ induced by the tautological torsor is a smooth surjective atlas. Let $t \colon T \to [X/G]$ be the morphism induced by $(T, \mathcal{P}, \pi_T)$. We restrict to an étale cover $\{f_i \colon T_i \to T\}$ and we take $t_i := f_i^* t$, so we can suppose without loss of generality that $\mathcal{P} \cong G_T = G \times_S T$. We want to prove that $X \times_{[X/G]} T \cong G_T$ and we are done. By Lemma 1.7.4, $X \times_{[X/G]} T \to T$ is a G_T-torsor. Moreover, as it is explained in Lemma 1.7.4, item 1., we have the natural map $\pi \colon X \times_{[X/G]} T \to X \times_S T$ is a morphism of G_T-torsors over the scheme T. However, a morphism of torsor is always an isomorphism, and this concludes the proof. □

When we are in the setting of Proposition 1.5.4, i.e. the morphism $G \to S$ is affine, this Corollary follows.

Corollary 1.7.6. If G/S is an affine smooth group scheme, the algebraic stack $[X/G]$ can be described as the stack over (Sch/S) whose objects are triples (T, P, ρ, π) where:

- T is an S-scheme and $(P \to T)$ is a principal G_T-bundle on T,
- $\pi \colon P \to X \times_S T$ is a G_T-equivariant map of T-schemes.

Morphisms are pairs $(f, g) \colon (T, P, \rho, \pi) \to (T', P', \rho', \pi')$ where $f : T \to T'$ is an S-morphism of schemes and $g \colon P \to P' \times_{T'} P$ is a G_T-equivariant morphism such that the induced diagram

$$\begin{tikzcd} P \arrow[rr, "g"] \arrow[dr, "\pi"] & & P' \times_{T'} T \arrow[dl, "f^*\pi'"] \\ & X \times_S T & \end{tikzcd}$$

commutes.

Definition 1.7.7 ($\mathbf{B}G$)**.** If G/S is a smooth group scheme, the *classifying stack* of G is the stack quotient $[S/G]$ where G acts trivially on S. It is denoted $\mathbf{B}G$.

These stacks are crucial for example in the study of Deligne-Mumford stacks with coarse moduli spaces, since one can prove that they are locally quotient stacks, see Theorem 1.8.4 in the following section.

The quotient stacks are easy to study; in particular, it is a pleasant surprise that the stack of elliptic curves is indeed a quotient stack, see Theorem 2.3.5 in the next chapter.

By the way, not all the algebraic stacks are quotient of schemes by group actions, hence we are interested in proving a criterion for when an algebraic stack is a global quotient, and the rest of the section is devoted to this.

We are now ready to state a criterion for an algebraic stack to be equivalent to a quotient stack $[X/G]$. As always in this thesis, we suppose that G is affine.

Theorem 1.7.8. Let $\rho\colon \mathcal{X} \to S$ be a stack. Let $x\colon X \to \mathcal{X}$ be a G-invariant map of stacks. Suppose that G is affine. If for all $t\colon T \to \mathcal{X}$

$$(pr_2\colon X \times_{\mathcal{X}} T \to T, \rho_T)$$

is a principal G_T-bundle, then $\mathcal{X}$ and $[X/G]$ are equivalent as stacks. In particular, there exists a 2-commutative diagram

$$\begin{tikzcd} X \arrow[rr, "x"] \arrow[dr] & & \mathcal{X} \arrow[dl, "\cong"] \\ & {[X/G]} & \end{tikzcd},$$

where $X \to [X/G]$ is the covering map defined by the tautological torsor

$$(X, (G \times_S X \to X), (\chi, q_2))$$

as in the proof of Theorem 1.7.5.

Proof. First of all, we want to define the morphism of fibered categories over S with the étale topology

$$F\colon \mathcal{X} \to [X/G].$$

For what concerns the objects, F associates to an object $T \to \mathcal{X}$ the triple

$$(T, (X \times_{\mathcal{X}} T \to T, \rho_T), \pi).$$

About the arrows, F associates to a morphism $f\colon T \to V$ over $\mathcal{X}$ the pair (f, h) where π and h are the maps defined in the Lemma 1.7.4. Thanks to the properties of π and h described in Lemma 1.7.4, F is well-defined.

Claim: F is an equivalence.

To check this, it is enough to prove that for each T/S

$$F_T\colon \mathcal{X}(T) \to [X/G](T)$$

is fully faithful and essentially surjective.

(Step 1) It is clear that the couple (f, h) determines uniquely f, hence F_T is faithful.

(Step 2) In this step we prove that F_T is full.

Fix two elements $t\colon T \to \mathcal{X}$ and $t'\colon T \to \mathcal{X}$, and consider a $[X/G](T)$-morphism $(id_T, h)\colon F_T(t) \to F_T(t')$. We use the notation ' to denote all the objects related to t'. We have to find a natural isomorphism $\beta\colon t \to t'$. By the commutativity of

$$\begin{tikzcd} X \times_{\mathcal{X}} T \arrow[rr, "h"] \arrow[dr, "\pi"] & & T \times_T (X \times_{\mathcal{X}} T) \arrow[dl, "\pi'"] \\ & X \times_S T, & \end{tikzcd}$$

we get

$$h(z, y, \sigma) = (y, (z, y, \sigma'))$$

and, in particular, an isomorphism $\sigma' \circ \sigma^{-1}\colon t(y) \to x(z) \to t'(y)$. Étale locally $\{T_i \to T\}_{i \in I}$ in T, the composition $e_{G_{T_i}}\colon T_i \to G_{T_i}$ with the isomorphisms given by a fixed section $p_i\colon G_{T_i} \to X \times_{\mathcal{X}} T_i$ insures a section $s_i\colon T_i \to X \times_{\mathcal{X}} T_i$ of the second projection. For all $y \in T_i$ set $\beta_y := \sigma'_i \circ \sigma_i^{-1}$, where the index i represents $x_i \in X$, i.e. we have $s_i(y) = (x_i, y, \sigma_i)$ and $h(x_i, y, \sigma_i) = (y, (x_i, y, \sigma'_i))$.

On the intersections T_{ij} there exists a unique $g \in G_{T_{ij}}$ such that $s_i|_{T_{ij}} = g \cdot s_j|_{T_{ij}}$. If we evaluate in $y \in T_{ij}$ we get

$$h(x_i, y, \sigma_i) = h(g \cdot (x_j, y, \sigma_j))$$

from which

$$h(\chi(g,x_j),y,\sigma_j\circ\alpha_{(g,x_j)}) = h(g\cdot(x_j,y,\sigma_j)) = \\ g\cdot h(x_j,y,\sigma_j) = g\cdot(y,(x_j,y,\sigma_j')) = \\ (y,(\chi(g,x_j),y,\sigma_j'\circ\alpha_{(g,x_j)}).$$

The second equality is true for the G_T-equivariance. Here α is the map that follows from the definition of invariance as before. Therefore,

$$\sigma_i'\circ\sigma_i^{-1} = (\sigma_j'\circ\alpha_{(g,x_j)})\circ(\sigma_j\circ\alpha_{(g,x_j)})^{-1} = \sigma_j'\circ\sigma_j^{-1}.$$

So we have a global map $\beta\colon t\to t'$, where

$$\beta_y\circ\sigma'\circ\sigma^{-1}\colon t(y)\to t'(y)$$

belongs to $\underline{Isom}_{\mathcal{X}(T)}(t(y),t'(y))$. By its construction, β behaves well on the arrows, that is it makes the top square in the diagram

$$\begin{tikzcd}
Y_1 \arrow[rr, "\beta_{y_1}"] \arrow[dr, "\Phi"] \arrow[ddr, "y_1"'] & & Y_1 \arrow[dr, "\Phi"] \arrow[ddr, "y_1"'] & \\
& Y_2 \arrow[rr, "\beta_{y_2}"] \arrow[d, "y_2"] & & Y_2 \arrow[d, "y_2"] \\
& T \arrow[rr, "id_T"] & & T
\end{tikzcd}$$

commute, and so it is a natural transformation. This complete the proof that the functor F_T is full.

(Step 3) In this step we prove that F_T is essentially surjective.

Given an element

$$(T,(\gamma\colon P\to T,\rho),\pi)\in[X/G],$$

we take a trivializing cover $\{T_i\to T\}_{i\in I}$ such that sections s_i of γ exist. For each i, we get a map of S-stacks

$$t_i := x\circ\pi_1\circ\pi\circ s_i\colon T_i\to\mathcal{X},$$

where π_1 is as in the statement of Lemma 1.7.4. As before, we have (unique) elements g of G_{ij} such that $s_i|_{T_{ij}} = g\cdot s_j|_{T_{ij}}$. By G_T-equivariance of $\pi_1\circ\pi$, if we evaluate in $y\in T_{ij}$, we get that $t_i|_{T_{ij}}(y)$ is equal to

$$x\circ\pi_1\circ\pi(g\cdot s_j|_{T_{ij}}(y)) = x(g\cdot(\pi_1\circ\pi\circ s_j|_{T_{ij}}(y))) = \alpha^{-1}_{(g,x_j)}(x\circ\pi_1\circ\pi_P\circ s_j|_{T_{ij}}(y)),$$

where $x_j = \pi_1\circ\pi\circ s_j|_{T_{ij}}(y)$. To be more explicit, for all i,j we have the isomorphism

$$\alpha_{(g,x_j)}\colon t_j|_{T_{ij}}(y)\to t_i|_{T_{ij}}(y).$$

By the condition of associativity of α and the uniqueness of the g's, we get descent data for $\mathcal{X}$: being a stack, the descent is effective and t_i glue to an S-map $t\colon T\to\mathcal{X}$. Furthermore, the solid diagram

$$\begin{tikzcd}
P \arrow[rrd, bend left, "\gamma"] \arrow[ddr, bend right, "\pi_1\circ\pi"'] & & \\
& X\times_{\mathcal{X}} V \arrow[r, dashed, "pr_2"] \arrow[d, dashed, "pr_1"'] & T \arrow[d, "t"] \\
& X \arrow[r, "x"] & \mathcal{X}
\end{tikzcd}$$

2-commutes: indeed, if we evaluate in $\in P$, locally expressed as $z_i\in P_i$, one gets

$$t_i\circ\gamma(z_i) = x\circ\pi_1\circ\pi\circ s_i\circ\gamma(z_i).$$

Moreover, if we consider the unique element $g_i\in G_i$ such that $s_i\circ\gamma(z_i) = g_i\cdot z_i$ then

$$t_i\circ\gamma(z_i) = x(g_i\cdot(\pi_1\circ\pi(z_i))) = \alpha^{-1}_{(g_i,\pi_1\circ\pi(z_i))}(\pi_1\circ\pi(z_i)).$$

Therefore, again since α is associative and from the uniqueness of the g_i's, the maps $\alpha_{(g_i,\pi_1\pi(z_i))}$'s glue to a global isomorphism

$$\alpha^{-1}_{(g_z,\pi_1\pi(z))}\colon x(\pi_1\pi(z)) \to t\circ\gamma(z)$$

and this is the natural isomorphism that makes the previous solid diagram 2-commute. By the property of fiber product we get an $\mathcal{X}$-map

$$h\colon P \to X\times_{\mathcal{X}} T.$$

We claim that (id_T, h) gives an isomorphism between

$$(T,(\gamma\colon P\to T,\rho),\pi)\to(T,(pr_2\colon X\times_{\mathcal{X}}T\to T,\rho_T),\pi_T).$$

Since both π and $\pi_T\circ h$ complete the diagram

$$\begin{tikzcd}
P \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "\pi_1\circ\pi"'] & & \\
& X\times_S V \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & T \arrow[d, "\rho\circ t"] \\
& X \arrow[r, "\rho\circ x"] & S,
\end{tikzcd}$$

the universal property of the fiber product $X\times_S T$ implies that the diagram

$$\begin{tikzcd}
P \arrow[rr, "h"] \arrow[dr, "\pi"'] & & X\times_{\mathcal{X}} T \arrow[dl, "\pi_T"] \\
& X\times_S T &
\end{tikzcd}$$

commutes. Finally, for the G_T-equivariance of h, we consider the square

$$\begin{tikzcd}
G_T\times_T P \arrow[r, "id_G\times h"] \arrow[d, "\rho"'] & G_T\times_T (X\times_{\mathcal{X}} T) \arrow[d, "\rho_T"] \\
P \arrow[r, "h"] & X\times_{\mathcal{X}} T.
\end{tikzcd}$$

For all $z\in P$, putting $z=\pi_1\pi(z)$,

$$\begin{tikzcd}
(g,z) \arrow[d, mapsto, "\rho"] & \\
\rho(g,z) \arrow[r, mapsto, "h"] & (\pi_1\pi(\rho(g,z)),\gamma(\rho(g,z)),\alpha^{-1}_{(g'_{\rho(g,z)},\pi_1\pi(\rho(g,z)))})
\end{tikzcd}$$

Observe that we have the equality

$$(\pi_1\pi(\rho(g,z)),\gamma(\rho(g,z)),\alpha^{-1}_{(g'_{\rho(g,z)},\pi_1\pi(\rho(g,z)))}) = (\chi(g,z),\gamma(z),\alpha^{-1}_{g'_{\rho(g,z)},\chi(g,z))})$$

which follow from the G_T-equivariance of $\pi_1\pi$ and from the G_T-invariancy of γ. On the other hand,

$$\begin{tikzcd}
(g,z) \arrow[r, mapsto, "id_{G_T}\times h"] & (g,z,\gamma(z),\alpha^{-1}_{(g'_z,z)}) \arrow[d, mapsto, "\rho_T"] \\
& (\chi(g,z),\gamma(z),\alpha^{-1}_{(g'_z,z)}\circ\alpha_{(g,z)}),
\end{tikzcd}$$

but since

$$g'_{z_i}\cdot z_i = s_i\circ\gamma(z_i) = s_i\circ\gamma(g\cdot z_i) = g'_{\rho(g,z_i)}(g\cdot z_i)$$

we have $g'_z = g'_{\rho(g,z)} \cdot g$ and by associativity of α

$$\alpha_{(g,z)} \circ \alpha_{g'_{\rho(g,z)},\chi(g,z))} = \alpha_{(g'_z,z)},$$

which means the equality

$$h \circ \rho = \rho_T \circ (id_{G_T} \times h).$$

Being a map between torsors, h is a G_T-isomorphism. Hence F_T is also essentially surjective.

(Step 4) In this step we prove the final part of the thesis of the Theorem.

Consider the morphism

$$(id_X, \delta)\colon (X, q_2\colon G \times_S X \to X, (\chi, pr_2)) \to (X, pr_2\colon X \times_{\mathcal{X}} X \to X, (\rho_X, \pi))$$

where the first is the trivial torsor and $\delta\colon G \times_S X \to X \times_{\mathcal{X}} X$ is the map which sends

$$(g, z) \mapsto (\chi(g, z), q_2(g, z), \alpha_{(g,z)}).$$

Since α is associative, δ is G_X-equivariant and it is obvious that it makes the diagram

$$\begin{tikzcd} G \times_S X \arrow[rr, "\delta"] \arrow[dr, "(\chi,q_2)"'] & & X \times_{\mathcal{X}} X \arrow[dl, "\pi"] \\ & X \times_S X & \end{tikzcd}$$

commute. This coincide exactly with the 2-commutativity of the diagram

$$\begin{tikzcd} X \arrow[rr, "x"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & {[X/G].} & \end{tikzcd}$$

This completes the proof.

□

1.8. Coarse Moduli Space

Definition 1.8.1. Let S be a scheme and $\mathcal{X}/S$ be an algebraic stack over S. A *coarse moduli space* for $\mathcal{X}$ is a morphism $\pi\colon \mathcal{X} \to X$ to an algebraic space over S such that:

1. π is initial for maps to algebraic spaces over S. That is, if $g\colon \mathcal{X} \to Z$ is a morphism from $\mathcal{X}$ to an algebraic space Z, there exists a unique morphism $f\colon X \to Z$ such that $g = f \circ \pi$.

2. For every algebraically closed field k the map $|\mathcal{X}(k)| \to X(k)$ is bijective, where $\mathcal{X}(k)$ denotes the set of isomorphism classes in $\mathcal{X}(k)$.

The main result on coarse moduli spaces is the following.

Theorem 1.8.2 (Keel-Mori)**.** Assume S is a scheme and that $\mathcal{X}$ is an algebraic stack locally of finite presentation over S with finite diagonal. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$. In addition:

1. Assume that S is locally noetherian; then X/S is locally of finite type, and if $\mathcal{X}/S$ is separated, then X/S is also separated.

2. π is proper, and the map $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

3. If $X' \to X$ is a flat morphism of algebraic spaces, then $\pi'\colon \mathcal{X}' := \mathcal{X} \times_X X' \to X'$ is a coarse moduli space for $\mathcal{X}'$.

Proof. See [Ols16], Theorem 11.1.2, or for more details the original article [KM97]. □

Moreover, we have that the formation of a coarse moduli space behaves well under flat base change.

Lemma 1.8.3. Let $\mathcal{X} \to X$ be a proper quasi-finite morphism, where $\mathcal{X}$ is a Deligne-Mumford stack and X a noetherian scheme. Let $X' \to X$ be a flat morphism of schemes, and denote $\mathcal{X}' := X' \times_X \mathcal{X}$.

1. If X is the moduli space of $\mathcal{X}$, then X' is the moduli space of $\mathcal{X}'$.
2. If $X' \to X$ is also surjective and X' is the moduli space of $\mathcal{X}'$, then X is the moduli space of $\mathcal{X}$.

Proof. See [AV02], Lemma 2.2.2. □

We make now a brief digression about separated Deligne-Mumford stacks and their local properties.

Theorem 1.8.4. Let S be a locally noetherian scheme and let $\mathcal{X}/S$ be a Deligne-Mumford stack locally of finite type and with finite diagonal. Let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. Let $\tilde{x} \to \mathcal{X}$ be a geometric point with image $\bar{x}\colon X$ in X. Let $G_{\tilde{x}}$ be the automorphism group of $\tilde{x}$, which is a finite group since $\mathcal{X}$ is Deligne-Mumford. Then there exists an étale neighbourhood $U_{\bar{x}} \to X$ of $\bar{x}$ ($U_{\bar{x}}$ can be chosen as the strictly henselian neighbourhood of $\bar{x}$) and there exists a finite $U_{\bar{x}}$-scheme $V_{\tilde{x}} \to U_{\bar{x}}$ with action of $G_{\tilde{x}}$, such that

$$\mathcal{X} \times_X U_{\bar{x}} \cong [V_{\tilde{x}}/G_{\tilde{x}}].$$

Proof. See [Ols16], Theorem 11.3.1. □

We will see a sort of generalization of the previous proposition in the case of tame stacks, see Theorem A.0.11 in the Appendix.

1.9. Quasi-coherent sheaves

It is possible to extend the notion of sheaf to the more general context of stacks. This is what we do in the rest of the chapter.

Definition 1.9.1 (Category $(Sch/\mathcal{X})$)**.** Let $\mathcal{X}/S$ be an algebraic stack. Define $(Sch/\mathcal{X})$ as the category containing as objects pairs (T, t) where T is a scheme over S and $t\colon T \to \mathcal{X}$ is a morphism of stacks over S. The morphisms $(T', t') \to (T, t)$ in $(Sch/\mathcal{X})$ are pairs (f, f^b) where $f\colon T' \to T$ is a morphism of S-schemes and $f^b\colon t' \to t \circ f$ is an isomorphism of functors $T' \to \mathcal{X}$.

We are now interested in defining a site on $\mathcal{X}$, called the Lis-Ét$(\mathcal{X})$ (*the lisse-étale site on* $\mathcal{X}$). This is the full subcategory of $(Sch/\mathcal{X})$ consisting of the pairs (T, t) where t is a smooth morphism, and a covering is a collection $\{(f_i, f_i^b)\colon (T_i, t_i) \to (T, t)\}$ such that $\{f_i : T_i \to T\}$ is an étale cover of T.

Definition 1.9.2. We define the sheaf $\mathcal{O}_{\mathcal{X}}$ on Lis-Ét$(\mathcal{X})$ by associating $\mathcal{O}_T(T)$ to (T, t).

Definition 1.9.3. A sheaf of $\mathcal{O}_{\mathcal{X}}$-modules on Lis-Ét$(\mathcal{X})$ is the data $(\{F_{(T,t)}\}, \{\Phi_{(f,f^b)}\})$. $F_{(T,t)}$ is an étale sheaf of $\mathcal{O}_T$ modules on T for each $(T, t) \in$ Lis-Ét$(\mathcal{X})$. For each morphism $(f, f^b)\colon (T', t') \to (T, t)$ in Lis-Ét$(\mathcal{X})$, $\Phi_{(f,f^b)}\colon f^*F_{(T,t)} \to F_{(T',t')}$ is a morphism of sheaves satisfying two condition:

- The $\Phi_{(f,f^b)}$ need to be compatible with composition, that is $\Phi_{(f,f^b)\circ(g,g^b)} = \Phi_{(g,g^b)} \circ g^*\Phi_{(f,f^b)}$.
- If we have (f, f^b) with f étale, then the map $\Phi_{(f,f^b)}$ has to be an isomorphism.

Morphism of sheaves

$$(\{F_{(T,t)}\}, \{\Phi_{(f,f^b)}\}) \to (\{E_{(T,t)}\}, \{\Psi_{(f,f^b)}\})$$

are a collection $\lambda_{(T,t)}\colon F_{(T,t)} \to E_{(T,t)}$ for each (T, t) such that the diagram

$$\begin{tikzcd} f^*F_{(T,t)} \arrow[r, "f^*\lambda_{(T,t)}"] \arrow[d, "\Phi_{(f,f^b)}"] & f^*E_{(T,t)} \arrow[d, "\Psi_{(f,f^b)}"] \\ F_{(T',t')} \arrow[r, "\lambda_{(T',t')}"] & E_{(T',t')} \end{tikzcd}$$

commutes.

Definition 1.9.4. We say that a sheaf on Lis-Ét$(\mathcal{X})$ of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F}$ is *cartesian* if for every morphism $(f, f^b)\colon (T', t') \to (T, t)$ the map of $\mathcal{O}'_T$-modules $\Phi_{(f,f^b)}\colon f^*\mathcal{F}_{(T,t)} \to \mathcal{F}_{(T',t')}$ is an isomorphism. A sheaf is *quasi-coherent* if $\mathcal{F}$ is cartesian and for every $(T, t) \in$ Lis-Ét$(\mathcal{X})$ the sheaf $\mathcal{F}_{(T,t)}$ is a quasi-coherent sheaf on T.

Definition 1.9.5. A sheaf of $\mathcal{O}_{\mathcal{X}}$-modules F is said to be invertible, or a line bundle, if each $F_{(T,t)}$ is an invertible sheaf on T. The group of line bundles over $\mathcal{X}$ is called Picard group and it is denoted as $Pic(\mathcal{X})$.

Observation 1.9.6. Alternatively, the Picard group can be defined as the sheaf cohomology group $H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})$, as in the classic setting. However, in this thesis we will not use this fact.

1.10. Quasi-coherent sheaves on BG

Schemes in this section are considered with the étale topology. In the following chapters we study pullbacks of coherent sheaves on $\mathbf{B}G$, and so it is useful to classify quasi-coherent shaves on $\mathbf{B}G$. We will prove the following.

Theorem 1.10.1. Let G a smooth group scheme over k. There is an equivalence of categories

$$\{\text{Quasi-coherent sheaves on } \mathbf{B}G\} \leftrightarrow \{\text{Representations of } G\}.$$

We follow the setting of [Sch].

Definition 1.10.2. Let V be a finite-dimensional k-vector space. Then we can think V as quasi-coherent sheaf on (Sch/k). We associate to $t\colon T \to k$ the $\mathcal{O}_T$-module $V \otimes_k \mathcal{O}_T$. We define $\underline{Aut}_k(V)$ as group scheme to be

$$\begin{aligned} \underline{Aut}_k(V)\colon (Sch/k)^{op} &\to (Grp) \\ (t\colon T \to k) &\mapsto Aut_T(V_T) = Aut_T(V \otimes_k \mathcal{O}_T), \end{aligned} \tag{1.11}$$

when we mean automorphism as the quasi-coherent sheaf over T.

A morphism $f\colon T \to T'$ over k is mapped to $\underline{Aut}_T(t^*V) \to Aut_{T'}(t'^*V)$, $\sigma \mapsto f^*\sigma$.

Definition 1.10.3. A morphism of functors $G \to \underline{Aut}_k(V)$, where V is a k-vector space, is said to be a *representation* of the group scheme G over the field k.

We denote with $(Repr)_k(G)$ the category of representations of group schemes G over k. The objects are the representations (V, Φ), i.e. V is a k-vector space and $\Phi\colon G \to \underline{Aut}_k(V)$ is the morphism of representation. The arrows $(V, \Phi) \to (W, \Psi)$ are G-equivariant maps $h\colon V \to W$, i.e., a k-linear map such that for all $g \in G(T)$, the diagram

$$\begin{tikzcd} V_T \arrow[r, "\Phi(g)"] \arrow[d, "h_T"] & V_T \arrow[d, "h_T"] \\ W_T \arrow[r, "\Psi(g)"] & W_T \end{tikzcd}$$

commutes, where h_T is the pullback of h via $T \to \{\text{pt}\}$.

To prove the equivalence of Theorem 1.10.1, we use an intermediate category. This category is extremely important, as we will see in Observation 1.10.8.

Definition 1.10.4. Let S be a scheme and G a smooth group scheme over S and X a scheme over S on which G acts via $\rho\colon G \times_S X \to X$. A *$G$-equivariant quasi-coherent sheaf* on X is a pair (F, σ) where F is a quasi-coherent sheaf on X and $\sigma\colon \rho^*F \to \mathrm{pr}_2^*F$ is an isomorphism such that for any S-scheme T and $g, g' \in G(T)$, the diagram of quasi-coherent sheaves on $T \times_S X$

$$\begin{tikzcd} \rho_g^*\rho_{g'}^*(\mathrm{pr}_2^*F) \arrow[r, "\rho_g^*\sigma_{g'}"] \arrow[d, "\cong"] & \rho_g^*(\mathrm{pr}_2^*F) \arrow[d, "\sigma_g"] \\ \rho_{g'g}^*(\mathrm{pr}_2^*F) \arrow[r, "\sigma_{g'g}"] & \mathrm{pr}_2^*F \end{tikzcd}$$

commutes, where

- $\mathrm{pr}_2 \colon T \times_S X \to X$ is the projection,
- ρ_g is $\mathrm{pr}_1 \times (\rho \circ (g \times \mathrm{id}_X)) \colon T \times_S X \to T \times_S X$ the map induced by the action, and
- $\sigma_g \colon \rho_g^* \mathrm{pr}_2^* F \to \mathrm{pr}_2^* F$ is the pullback of σ via map $g \times \mathrm{id}_X$.

Denote the category with object quasi-coherent G-equivariant sheaves on a scheme X by $(Qcoh^G(X))$. A morphism $(F, \sigma) \to (E, \tau)$ is a morphism $f \colon F \to E$ of quasi-coherent sheaves on X such that

$$\begin{tikzcd} \rho^* F \arrow[r, "\sigma"] \arrow[d, "\rho^* f"] & \mathrm{pr}_2^* F \arrow[d, "\mathrm{pr}_2^* f"] \\ \rho^* E \arrow[r, "\tau"] & \mathrm{pr}_2^* E \end{tikzcd}$$

commutes.

Observation 1.10.5. Notice that the definition given before is just an intricate way to say this:

A quasi-coherent sheaf F on S extends naturally to a functor

$$\begin{aligned} F \colon (Sch/S)^{op} &\longrightarrow (Set) \\ (f \colon T \to S) &\mapsto F(T) := (f^* F)(T) \end{aligned}$$

Each $F(T)$ has the structure of an $\mathcal{O}(T)$-module.

Then, an action of G on F is an $\mathcal{O}$-linear action of the functor

$$G \colon (Sch/S)^{op} \longrightarrow (Grp)$$

on F. In other words, for each $f \colon T \to S$ we have an action of the group $G(T)$ on the $\mathcal{O}(T)$-module $F(T)$, and this action is functorial in $T \to S$.

Now we state a Lemma that we will use in the proof of the Theorem 1.10.1.

Lemma 1.10.6. Let X, Y be S-schemes with G-actions $a_X \colon G \times_S X \to X$ and $a_Y \colon G \times_S Y \to Y$ respectively and let $f \colon Y \to X$ be a G-equivariant morphism of schemes. Let (F, σ) be a G- equivariant quasi-coherent sheaf on X, then $(f^* F, (id_G \times f)^* \sigma)$ is a G-equivariant quasi-coherent sheaf on Y.

Proof. Firstly, $f^* F$ is a quasi-coherent sheaf on Y. Then $(id_G \times f)^* \sigma$ is an isomorphism from

$$(id_G \times f)^* a_X^* F \cong (a_X \circ (id_G \times f))^* F = (f \circ a_Y)^* F \cong a_Y^* (f^* F)$$

to

$$(id_G \times f)^* \mathrm{pr}_X^* F \cong (\mathrm{pr}_X \circ (id_G \times f))^* F = (f \circ \mathrm{pr}_Y)^* F \cong \mathrm{pr}_Y^* (f^* F)$$

Let T a S-scheme and g, $g' \in G(T)$. If we pull back the commutative diagram on $T \times_S X$ via $(id_T \times f)$, the commutativity of the diagram of quasi-coherent sheaves on $T \times_S Y$ is immediate, hence we have the thesis. □

The next proposition is the reason for introducing G-equivariant quasi-coherent sheaves.

Proposition 1.10.7. Let G be a smooth group scheme over k. There is an equivalence of categories

$$(Qcoh^G(\mathrm{pt})) \leftrightarrow (Repr_k(G)). \tag{1.12}$$

Proof. Let $\rho \colon G \to \mathrm{pt}$ denote the trivial map. Let (F, σ) be an object of $(Qcoh^G(\mathrm{pt}))$, then we can see F as a k-vector space and $\sigma \in Aut_G(\rho^* F)$ (since in this case $\rho \equiv \mathrm{pr}$). It defines a scheme morphism $\rho_\sigma \colon G \to \underline{Aut_k(F)}$ sending a $g \in G(T)$ to $\sigma_g \in Aut_T(g^* \rho^* F) \cong Aut_T(t^* F)$ where $t \colon T \to k$. The map ρ_g is a group homomorphism. Actually $\sigma_{g'g} = \sigma_g \sigma_{g'}$ since the G-equivariance structure. Then $\rho_\sigma \colon G \to \underline{Aut_k(F)}$ is a representation.

Given a representation $\rho \colon G \to \underline{Aut_k(V)}$, the k-vector space V naturally gives a quasi-coherent sheaf on $\{pt\}$. Moreover, $\tilde{\rho} := \rho(\mathrm{id}_G)$ is an element of $Aut_G(\rho^* V)$ such that $G(T) \to Aut_T(V_T)$, $g \mapsto \rho(g) = g^* \tilde{\rho}$ is a group homomorphism since $\tilde{\rho}$ satisfies $(g'g)^* \tilde{\rho} = g^* \tilde{\rho} g'^* \tilde{\rho}$ for $g, g' \in G(T)$. Therefore $(V, \tilde{\rho})$ is a G-equivariant sheaf on pt.

We see now that the functor we have defined on the objects sends arrows into arrows. Let $f\colon (F,\sigma) \to (E,\tau)$ be a morphism in $Qcoh^G(pt)$, and consider the associated representation $\rho_g\colon G \to \underline{Aut_k(F)}$ and $\rho_\tau\colon G \to \underline{Aut_k(E)}$. Then we have

$$\begin{tikzcd} \rho^* F \arrow[r, "\sigma"] \arrow[d, "\rho^* f"] & \mathrm{pr}^* F \arrow[d, "\mathrm{pr}^* f"] \\ \rho^* E \arrow[r, "\tau"] & \mathrm{pr}^* E \end{tikzcd}$$

,

and so the induced linear map f satisfies for every k-scheme T and for all $g \in G(T)$ $\tau_g \circ f_T = f_T \circ \sigma_g$ obtained by pulling back along g. So we have indeed a morphism of representations.

For the other direction, given a map $h\colon V \to W$ between representations $\Phi\colon G \to \underline{Aut_k(V)}$ and $\Psi\colon G \to \underline{Aut_k(W)}$, it induces a morphism $(V, \Phi(e)) \to (W, \Psi(e))$. The first construction is the inverse of the other one, so $(Qcoh^G(\mathrm{pt}))$ is equivalent to $(Repr_k(G))$.

$\square$

One might wonder why we decide to prove Theorem 1.10.1 passing through this intermediate category. The reason is that, as we will see, the quasi-coherent sheaves over the quotient stack $[U/G]$ are exactly the G-equivariant sheaves over U, see Observation 1.10.8.

The following result, which is a particular case of the previous assertion, is exactly what we need to conclude the proof of Theorem 1.10.1.

Proof. By the previous proposition, it is enough to construct an equivalence

$$\{\text{Quasi-coherent sheaves on } \mathbf{B}G\} \leftrightarrow (Qcoh^G(\mathrm{pt})).$$

(Step 1) In this step, given an object (F,σ) of $(Qcoh^G(\mathrm{pt}))$, we construct a quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{B}G$ by giving data $\{\mathcal{F}_{(T,t)}, \Phi_{(f,f^b)}\}$.

(Step 1.1) Let (T,t) be an object in Lis-Ét$(\mathbf{B}G)$ and $\mathcal{P}$ the G-bundle on T defined by $t\colon T \to \mathbf{B}G$. We define $\mathcal{F}_{(T,t)}$ as follows. Let $f\colon \mathrm{pt} \to \mathbf{B}G$ be the morphism defined by the trivial bundle, then we have the pullback square

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d, "b"] & T \arrow[d, "t"] \\ \mathrm{pt} \arrow[r, "f"] & \mathbf{B}G \end{tikzcd}$$

This square is a pull back as it is proved in the proof of Theorem 1.7.5.

By Lemma 1.10.6, the pullback $(b^*F, (id_G \times b)^*)\sigma$ is again a G-equivariant quasi-coherent sheaf on $\mathcal{P}$. We will descend it along $\mathcal{P} \to T$. Write $a\colon G \times_k \mathcal{P} \to \mathcal{P}$ for the G-action on G-bundle $\mathcal{P}$, then the map $a \times \mathrm{pr}_2\colon G \times_k \mathcal{P} \to \mathcal{P} \times_k \mathcal{P}$ is an isomorphism. Therefore, the map τ defined as $\tau := ((a \times \mathrm{pr}_2)^{-1})^*(\mathrm{id}_G \times b)^*\sigma$ is an isomorphism $\mathrm{pr}_1^* b^* F \to \mathrm{pr}_2^* b^* F$. Moreover, τ satisfies cocycle condition, because of the G-equivariance structure on b^*F. Hence, (b^*F, τ) is a descent datum. Observe that the map $\mathcal{P} \to T$ is an fppf morphism, and so this descent datum is effective, so we obtain a quasi-coherent sheaf $\mathcal{F}_{(T,t)}$ on T.

(Step 1.2) Let $(f, f^b)\colon (T', t') \to (T,t)$ be a morphism in Lis-Ét$(\mathbf{B}G)$. The morphism (f, f^b) is the composition

$$(T', \mathcal{P}') \xrightarrow{(id_{T'}, f^b)} (T', f^*\mathcal{P}) \xrightarrow{(f,id)} (T, \mathcal{P}),$$

and so we will construct the isomorphism $\Phi_{(f,f^b)}\colon f^*F_{(T,t)} \to F_{(T,t)}$ via $\Phi_{(\mathrm{id}_{T'},f^b)} \circ f^*\Phi_{(f,\mathrm{id})}$. Consider $(id, f^b)\colon (T', \mathcal{P}') \to (T', \mathcal{P})$. Because G-bundles are locally trivial, we consider an étale cover $U \to T'$, where both bundles are trivial. An isomorphism of trivial G-bundles is given by multiplication m_g by a certain element $g \in G(U)$. In the diagram

$$\begin{tikzcd} G \times_k U \arrow[r, "m_g"] & G \times_k U \arrow[r] \arrow[d, "b"] & U \arrow[l, bend right, "e"'] \arrow[d] \\ & \mathrm{pt} \arrow[r] & \mathbf{B}G \end{tikzcd}$$

the square is a pullback square and e is the section given by the unit in G. The sheaves $\mathcal{F}_{(U,\mathcal{P})}$ and $\mathcal{F}_{(U,\mathcal{P}')}$ equal e^*b^*F and $e^*m_g^*b^*F$ respectively and we define isomorphism $\Phi_{(\mathrm{id}_U,m_g)}$ as

$$e^*m_g^*b^*F \cong (b\circ m_g\circ e)^*F \cong \rho_g^*F_U \to F_U \cong e^*b^*F.$$

By descent for an étale cover, this also defines $\Phi_(id_{T'}, f^b)$. Consider $(f,id)\colon (T', f^*\mathcal{P}) \to (T,\mathcal{P})$. We have the diagram

$$\begin{tikzcd} f^*\mathcal{P} \arrow[r] \arrow[d, "b'"] & T' \arrow[d, "f"] \\ \mathcal{P} \arrow[r] \arrow[d, "b"] & T \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathbf{B}G \end{tikzcd}$$

where all squares are pullback squares. By definition of $\mathcal{F}_{(T',f^*\mathcal{P})}$ its pullback to $f^*\mathcal{P}$ is defined by $(b\circ b')^*F$, but also the pullback of $f^*\mathcal{F}_{(T,\mathcal{P})}$ to $f^*\mathcal{P}$ is by commutativity of the upper square pulling back along b' of b^*F. Define $\Phi_{(f,id)}$ to be the natural isomorphism between $\mathcal{F}_{(T',f^*\mathcal{P})}$ and $f^*\mathcal{F}_{(T,\mathcal{P})}$.

(Step 1.3) Because we have mapped all morphisms to isomorphisms, $\mathcal{F}$ is cartesian and since all $\mathcal{F}_{(T,\mathcal{P})}$ are quasi-coherent, this data yields a quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{B}G$.

(Step 2) In this step, given a quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{B}G$ (i.e. the data of $\{\mathcal{F}_{(T,t)}, \Phi_{(f,f^b)}\}$), we provide a construction of a G-equivariant sheaf over (pt). Let $\tilde{G}$ be the trivial G-bundle on pt, and consider the quasi-coherent sheaf $F := \mathcal{F}_{pt,\tilde{G}}$ on pt. Let $\rho\colon G \to \mathrm{pt}$ be the trivial map. The objective is to construct an automorphism σ of ρ^*F such that $(F,\sigma) \in (Qcoh^G(\mathrm{pt}))$.

Consider $G\times_k G$ as a G-bundle on G via pr_2. Let $m\colon G\times_k G \to G\times_k G$, $(g,h)\mapsto (gh,h)$ denote this morphism of G-bundles over G. It provides an isomorphism $\Phi_{(id,m)}$ of $\mathcal{F}_{(G,G\times_k G)}$. Also, there is a morphism $(p,p^b)\colon (G, G\times_k G) \to (\mathrm{pt},\tilde{G})$ where p^b is an isomorphism induced by $(p\circ t)^* \cong t^*p^*$, giving an isomorphism $\Phi_{(p,p^b)}$. At this point, we define the isomorphism $\sigma \in Aut_G(p^*F)$ as the composition

$$p^*\mathcal{F}_{(\mathrm{pt},\tilde{G})} \xrightarrow{\Phi_{(p,p^b)}} \mathcal{F}_{(G,G\times_k G)} \xrightarrow{\Phi_{(id,m)}} \mathcal{F}_{(G,G\times_k G)} \xrightarrow{\Phi^{-1}_{(p,p^b)}} p^*\mathcal{F}_{(\mathrm{pt},\tilde{G})}.$$

For a k-scheme T and g, $g' \in G(T)$, we have for $\sigma_g = g^*\sigma$ that $\sigma_g \circ \sigma'_g = \sigma_{g'g}$, because $g^*\Phi_{(id,m)}$ is the map induced by m_g. Hence, σ gives G-equivariance structure for F.

Step 3: For a morphism $\lambda_{(T,t)}$ between quasi-coherent sheaves on $\mathbf{B}G$ defined by $\{\mathcal{F}_{(T,t)}, \Phi_{(f,f^b)}\}$ and $\{\mathcal{E}_{(T,t)}, \Psi_{(f,f^b)}\}$ we take the $(\mathrm{pt},\tilde{G})$ component to define the morphism $\lambda_{(\mathrm{pt},\tilde{G})}\colon \mathcal{F}_{(\mathrm{pt},\tilde{G})} \to \mathcal{E}_{(\mathrm{pt},\tilde{G})}$ of G-equivariant quasi-coherent sheaves. Vice versa, given a morphism $h\colon (F,\sigma) \to (E,\tau)$ of G-equivariant quasi-coherent sheaves, we built $\lambda_{(T,\mathcal{P})}$ by descending b^*h between quasi-coherent sheaves on $\mathcal{P}$. These constructions give an equivalence. □

Observation 1.10.8. At the same way, let $[U/G]$ be a quotient stack. It is easy to prove that there is an equivalence

$$\{\text{Quasi-coherent sheaves on } [U/G]\} \leftrightarrow (Qcoh^G(\mathrm{U})).$$
$$\mathcal{F} \mapsto (\mathcal{F}_u, \sigma_{id_G\times\rho}\colon \rho^*(\mathcal{F}_u) \to pr^*(\mathcal{F}_u)),$$

where $u\colon U \to [U/G]$ is the tautological torsor. For the proof we use exactly the same argument as in the previous proof. To streamline the presentation, the details are omitted.

Definition 1.10.9. Let $\bar{G}$ denote the *group of characters* $\chi\colon G \to \mathbb{G}_{m,R}$ consisting of $Spec(R)$-morphisms of group schemes, with $Spec(R)$ connected.

We can generalize the previous result in the case of the quotient stack $[Spec(R)/G]$, where G is a R-group scheme acting trivially on the spectrum of the ring R.

Corollary 1.10.10. If G is a smooth R-group scheme acting trivially on R, where $Spec(R)$ is a connected scheme, then we have the isomorphism

$$Pic([Spec(R)/G]) \cong Pic(R) \times \bar{G}.$$

CHAPTER 2

The Stack of the Elliptic Curves

In this chapter we review some basic facts about elliptic curves and after we analyze the stack of elliptic curves $\mathcal{M}_{1,1}$.

Definition 2.0.1 (Curve)**.** A *curve* over an algebraically closed field k is an integral regular scheme of dimension 1, proper over k. The *genus* of a curve C over k is defined to be $dim_k H^1(C, \mathcal{O}_C)$.

Note that implies in particular that the morphism $C \to Spec(k)$ is smooth.

Definition 2.0.2. A pair (E, O), where E is a genus 1 curve over $k = \bar{k}$ together with a distinguished k-valued point $O \in E$, is called *elliptic curve.*

Definition 2.0.3. An *elliptic curve over an arbitrary scheme* S is a smooth proper morphism $\rho\colon E \to S$ together with a chosen section $e\colon S \to E$ so that the pullback of (E, e) to any geometric fiber is an elliptic curve.

Definition 2.0.4. Let $\mathcal{M}_{1,1}$ denote the fibered category over $Spec(\mathbb{Z})$ whose objects are collections of data $(S, (E, e))$, where S is a scheme and (E, e) is an elliptic curve over S. A morphism

$$(S', (E', e')) \to (S, (E, e))$$

is a pair of morphisms (f, g) fitting into a cartesian diagram

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

such that $g \circ e' = e \circ f$. The projection

$$\begin{aligned} \mathcal{M}_{1,1} &\to (Schemes) \\ (S, (E, e)) &\mapsto S \end{aligned}$$

makes $\mathcal{M}_{1,1}$ a fibered category over the category of schemes.

Definition 2.0.5. Let S be a scheme. We denote with $\mathcal{M}_{1,1,S} := \mathcal{M}_{1,1} \times S$ the fibered category over the category of S-schemes. Explicitly, for each S-scheme T, $\mathcal{M}_{1,1,S}(T)$ is the groupoid of the elliptic curves over T.

The main theorem of this chapter is the following.

Theorem 2.0.6. The fibered category $\mathcal{M}_{1,1}$ is a separated Deligne-Mumford stack, with respect to the fpqc topology, of finite type over $Spec(\mathbb{Z})$.

We prove this theorem in sections below. Observe that we focus on $\mathcal{M}_{1,1}$ because, once we have deduced its properties, the analogous result for $\mathcal{M}_{1,1,S}$ (with S a general scheme) immediately follows.

Corollary 2.0.7. $\mathcal{M}_{1,1,S}$ is a separated Deligne-Mumford stack, with respect to the fpqc topology, of finite type over S.

Proof. Obvious, since all these properties are satisfied by $\mathcal{M}_{1,1}$ in light of Theorem 2.0.6 and are stable under base change. $\square$

2.1. Elliptic curves over algebraically closed field

From now on throughout this section we assume that $k = \bar{k}$ is an algebraically closed field. In this section we review the classical theory for elliptic curves over an algebraically closed field, following the approach of Silverman's The Arithmetic of Elliptic Curves ([Sil09]), mostly for what concerns Chapter 3. This is just a reminder, so we omit the majority of proofs. There are two reason for doing so. First of all, we need to fix once and for all the notation about elliptic curves we use in all the thesis (for example in Chapter 3). Secondly, we want to emphasize the analogies with the more general case discussed in Section 2.3.

Let $\mathbb{P}^2_k$ be the projective plane with coordinates $[x\,,\,y\,,\,z]$ and let E be the closed subscheme defined by the equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3. \tag{2.1}$$

This is well known as the *Weierstrass equation.* Here $O = [0, 1, 0]$ is the base *base point* and the coefficients $a_1, \ldots, a_6 \in \bar{k}$.

To ease notation, we generally write the Weierstrass equation for our elliptic curve using non-homogeneous coordinates $x = X/Z$ and $y = Y/Z$,

$$E\,:\,y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{2.2}$$

always remembering that there is an extra point $O = [0, 1, 0]$ not belonging to this affine hyperplane.

Define

$$\begin{aligned} b_2 &= a_1^2 + 4a_2,\\ b_4 &= a_1a_3 + 2a_4,\\ b_6 &= a_3^2 + 4a_6,\\ b_8 &= -a_1a_3a_4 - a_4^2 + a_1^2a_6 + a_2a_3^2 + 4a_2a_6, \end{aligned} \tag{2.3}$$

and the discriminant

$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6. \tag{2.4}$$

We also define quantities

$$\begin{aligned} c_4 &= b_2^2 - 24b_4,\\ c_6 &= -b_2^3 + 36b_2b_4 - 216b_6,\\ j &= c_4^3/\Delta\\ \omega &= \frac{dx}{2y + a_1x + a_3} = \frac{dy}{3x^2 + 2a_2x + a_4 - a_1y}. \end{aligned} \tag{2.5}$$

One easily verifies they satisfy the relations

$$\begin{aligned} 4b_8 &= b_2b_6 - b_4^2,\\ 1728\Delta &= c_4^3 - c_6^2. \end{aligned} \tag{2.6}$$

Definition 2.1.1. The quantity Δ is the *discriminant* of the Weierstrass equation, the quantity j is the *j-invariant* of the curve, and ω is the *invariant differential* associated to the Weierstrass equation.

Proposition 2.1.2. 1. The curve given by a Weierstrass equation satisfies:

- It is non singular if and only if $\Delta \neq 0$.
- It has a node if and only if $\Delta = 0$ and $c_4 \neq 0$.
- It has a cusp if and only if $\Delta = c_4 = 0$.

Moreover if the curve is singular there is exactly one singular point.

2. Two elliptic curves are isomorphic over $\bar{k}$ if and only if they both have the same j-invariant.
3. Let $j_0 \in \bar{k}$. There exists an elliptic curve defined over $K(j_0)$ whose j-invariant is equal to j_0.

Proof. See Proposition 3.1.4. of [Sil09]. □

Proposition 2.1.3. Let E be a curve defined by a Weierstrass equation. Then the associated invariant differential ω has neither zeros nor poles, i.e. $div(\omega) = 0$.

Proof. See Proposition 3.1.5. of [Sil09]. □

In order to connect the notion of elliptic curve with the previous quantities defined, we use the Riemann-Roch theorem to show that every elliptic curve can be written as a plane cubic, and conversely, every smooth Weierstrass plane cubic curve is an elliptic curve.

We give the whole proof, so the reader will be able to appreciate the similarities with the more general case expressed by the theorem 2.3.4.

Theorem 2.1.4. Let E be an elliptic curve defined over k.

1. There exist functions x, $y \in k(E)$ such that the map
$$\begin{aligned} \Phi\colon E &\to \mathbb{P}^2 \\ \Phi &= [x, y, 1], \end{aligned} \tag{2.7}$$
gives an isomorphism of E/k onto a curve given by a Weierstrass equation
$$\mathcal{C}:\ Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$
with coefficients $a_1, \ldots, a_n \in k$ and satisfying $\Phi(O) = [0, 1, 0]$.
2. Any two Weierstrass equations for E as in (1) are related by a linear change of variables of the form
$$\begin{aligned} X &= u^2X' + r, \\ Y &= u^3Y' + sX' + t. \end{aligned} \tag{2.8}$$
with $u \in k^*$ and $r, s, t \in k$.
3. Conversely, every smooth cubic curve $\mathcal{C}$ given by a Weierstrass equation as in (1), is an elliptic curve defined over k with base point $O = [0, 1, 0]$.

Proof. **(1)** We look at the vector space $\mathcal{L}(n(O))$ for $n = 1, 2, \ldots$. By the Riemann-Roch theorem, since $g = 1$, we have
$$l(n(O)) = dim\mathcal{L}(n(O)) = n \quad \text{for all } n \geq 1. \tag{2.9}$$
Thus we can choose functions x, $y \in k(E)$ so that $\{1, x\}$ is a basis for $\mathcal{L}(2(O))$ and so that $\{1, x, y\}$ is a basis for $\mathcal{L}(3(O))$. Note that x must have a pole of exact order 2 at O (if the order had been at most 1, then the vector space $\mathcal{L}((O))$ would have had dimension 1, absurd). Similarly, y must have a pole of exact order 3 at O.

Now we observe that $\mathcal{L}(6(O))$ has dimension 6, but it contains the seven functions 1, x, y, x^2, xy, y^2, x^3. It follows that there is a linear relation
$$A_1 + A_2x + A_3y + A_4x^2 + A_5xy + A_6y^2 + A_7x^3 = 0, \tag{2.10}$$
where the coefficients belongs to k. Note that $A_6A_7 \neq 0$, since otherwise every term would have a pole at O of a different order, and so all the A_j's would vanish. Replace x and y by $-A_6A_7x$ and $A_6A_7^2y$ respectively, and dividing by $A_6^3A_7^4$, we get a cubic equation in Weierstrass form. This gives a map
$$\begin{aligned} \Phi\colon E &\to \mathbb{P}^2 \\ \Phi &= [x, y, 1], \end{aligned}$$
whose the image C lies in the locus described by a Weierstrass equation.

Note that $\Phi\colon E \to C$ is a morphism since for a smooth curve a rational map is defined at every point (see Proposition 2.2.1 of [Sil09]) and that is surjective, since every morphism of curves is constant or surjective (see Proposition 2.2.3 of [Sil09]). Further we have $\Phi(O) = [0,1,0]$, since y has an higher-order pole than x at the point O.

The next step is to show that the map $\Phi\colon E \to C \subset \mathbb{P}^2$ has degree-one, or equivalently, to show that $k(E) = k(x,y)$. Consider the map $[x,1]\colon E \to \mathbb{P}^1$. Since x has double pole at O and no other poles, we know this map has degree 2. Thus $[k(E) : k(x)] = 2$. Similarly, the map $[y,1]\colon E \to \mathbb{P}^1$ has degree 3, and so $[k(E) : k(x,y)] = 3$. Therefore $[k(E) : k(x,y)]$ divides both 2 and 3, so it must be equal to 1.

Next we show that C is smooth. Suppose that C is singular. Then for Proposition 2.2.2 of [Sil09], there is a rational map $\Psi\colon C \to \mathbb{P}^1$ of degree one. It follows that the composition $\Psi \circ \Phi\colon E \to \mathbb{P}^1$ is a map of degree one between smooth curves, so is an isomorphism. This contradicts the fact that E has genus 1. Therefore, C is smooth and the map Φ is an isomorphism between E and C.

(2) Let $\{x,\, y\}$ and $\{x',\, y'\}$ be two sets of Weierstrass coordinate functions on E. Then x and x' have poles of order 2 at O, and y and y' have poles of order 3 at O. Hence $\{1,\, x\}$ and $\{1,\, x'\}$ are both bases for $\mathcal{L}(2(O))$, and similarly $\{1,\, x,\, y\}$ and $\{1,\, x',\, y'\}$ are both bases for $\mathcal{L}(3(O))$. Thus there are constants

$$u_1, u_2 \in k^* \;\text{ and }\; r, s_2, t \in k$$

such that

$$\begin{cases} x &= u_1 x' + r, \\ y &= u_2 y' + s_2 x' + t. \end{cases}$$

Since both (x,y) and (x',y') satisfy Weierstrass equations in which the Y^2 and the X^3 terms have coefficient 1, we have $u_1^3 = u_2^2$. Letting $u = \frac{u_2}{u_1}$ and $s = \frac{s_2}{u^2}$ puts the change of variables formula into the desired form.

(3) Let E be given by a Weierstrass equation. We have seen in Proposition 2.1.3 that the differential $\omega \in \Omega_E$ has neither zeros nor poles, so $div(\omega) = 0$. The Riemann-Roch theorem then tells us that

$$2genus(E) - 2 = deg(div(\omega)) = 0,$$

so E has genus 1, and taking $[0,1,0]$ has the base point makes E into an elliptic curve. □

2.2. The Stack $\mathcal{M}_{1,1}$

In this section we prove that $\mathcal{M}_{1,1}$ is a stack in the fpqc topology on (Sch). Before doing this, we need some preliminaries. We start developing an important tool for reducing problems about elliptic curve over arbitrary basis to elliptic curves over noetherian schemes.

Lemma 2.2.1. The following results hold.

- Let $\rho\colon E \to Spec(R)$ be quasi-compact, separated, smooth morphism of relative dimension 1. Then it is pulled back from such a morphism over noetherian affine scheme $\rho_0\colon X_0 \to Spec(R_0)$.
- If ρ in addition has a section $e\colon Spec(R) \to E$, it can be arranged to be pulled back from a section $Spec(R_0) \to X_0$.
- Let $(E, \rho\colon E \to Spec(R), e)$ be an elliptic curve over an affine scheme. Then it is pulled back from an elliptic curve over noetherian affine scheme.

Proof. **(1)** We can write the ring R as a filtered colimit over a poset I of subrings R_i which are of finite type over $\mathbb{Z}$, so in particular noetherian, and thus $Spec(R) \cong \varprojlim Spec(R_i)$, see [GW10] Proposition 10.53.

So we are in the situation of [Sta22] Tag $[01ZM]$. By possibly restricting to a cofinal subset I we may assume I to have an initial object $0 \in I$ and we might assume to have a morphism $\rho_0\colon X_0 \to Spec(R_0)$ of finite presentation such that the morphism ρ is the pullback of ρ_0 along the projection $Spec(R) \to Spec(R_0)$. Now we are in the situation of [Sta22] Tag $[0C0C]$, so by possibly restricting to a cofinal subset I again, we may assume that ρ_0 is smooth. Moreover, by restricting again we can assume that ρ_0 is separated too.

(2) Using [Sta22] Tag [$01ZM$] for morphisms, we obtain also a map $e_0\colon Spec(R_0) \to X_0$ of schemes over $Spec(R_0)$, so a section of ρ_0, whose pullback to $Spec(R)$ is precisely e.

(3) Now we can apply [Sta22] Tag [0204] to conclude that we also may assume ρ_0 to be proper. Since its pullback ρ is of relative dimension 1, so is ρ_0. Our next goal is to show that the geometric fibers of ρ_0 are connected. We can use [Sta22] Tag [$0CC1$] to conclude that the image of $Spec(R)$ is dense in $Spec(R_0)$ since R_0 is a subring of R. Consider now any geometric point $Spec(k) \to Spec(R_0)$. Since R_0 is a noetherian ring, this implies that there is a connected open U containing the image of $Spec(k)$, by [Sta22] Tag [$04MF$]. Since it is open, by density, the image of some point (and thus also of a geometric point) $Spec(L) \to R$ belongs to U, so we have a commutative diagram

$$\begin{tikzcd} Spec(k) \arrow[r] & U \arrow[r, "\subset"] & Spec(R_0) \\ & Spec(L) \arrow[u] \arrow[r] & Spec(R). \arrow[u] \end{tikzcd}$$

We mean that L is some algebraically closed field. The pullback $\rho_Y\colon Y := X_0 \times_{Spec(R_0)} U \to U$ of ρ_0 is now again a proper morphism with a section, smooth of relative dimension 1 over a noetherian connected scheme. Over the geometric point $Spec(L) \to U$, it is connected since by assumption $(X_0 \times_{Spec(R_0)} U) \times_U Spec(L) \cong E \times_{Spec(R)} Spec(L)$ is an elliptic curve over L.

Now consider the Stein factorization $Y \longrightarrow Y' \xrightarrow{g} U$ of the proper morphism of noetherian scheme $\rho_Y\colon Y \to U$. By [Sta22] Tag [$034E$] being reduced is local in the smooth topology, and therefore the fibers of the smooth map ρ_y are geometrically reduced. We can in particular deduce that $g\colon Y' \to U$ is étale, by [FGI+05], Proposition 8.5.16. Thus also $Y' \times_U Spec(L) \to Spec(L)$ is étale, and so $Y' \times_U Spec(L)$ is a finite disjoint union of copies of $Spec(L)$. Next, by Zariski's connectedness Theorem, the fibers of $Y \to Y'$ are non-empty and connected. Since being non-empty is compatible with faithfully flat base change, we conclude that also the fibers of $Y \times_U Spec(L) \to Y' \times_U Spec(L)$ are non-empty. Since $Y \times_U Spec(L)$ is connected, we conclude that $Y' \times_U Spec(L) \cong Spec(L)$. Moreover, by definition of Stein factorization, the map $g\colon Y' \to U$ is finite, thus we can apply [Sta22] Tag [$02KG$] to conclude that $g_*\mathcal{O}_Y$ is a vector bundle on U. Employing cohomology and base change and the fact that the pullback of g to $Spec(L)$ is an isomorphism, we conclude that $g_*\mathcal{O}_{Y'}$ is a vector bundle of rank 1 on the connected scheme U. Thus, also the pullback of g to $Spec(k)$ is an isomorphism, and using Zariski's connectedness theorem once again implies that the fibers $Y \to U$ are geometrically connected.

So we know that the geometric fibers of ρ_Y are geometrically connected. Again, by [Har13], Example 3.10.0.3, the geometric fibers are regular of dimension 1. Thus they are in particular normal ([Sta22] Tag [0569]) and thus ([Sta22] Tag [$033M$]) integral since we have already shown that they are connected.

This implies that the geometric fibers of ρ_Y (and thus also of ρ_0) are curves. We still need to determine the genus of these curves. With the notation above, we know that the curve $Y \times_U Spec(L) \to Spec(L)$ has genus 1. As explained in [Gro67] Theorem 7.9.4, the Euler characteristic of the geometric fibers is constant on the connected scheme U. By [**?**] Corollary 3.3.21, we know that $H^0(Y \times_U Spec(k), \mathcal{O}_{Y \times_U Spec(k)})$ is 1-dimensional, and by Grothendieck's vanishing theorem cohomology groups in dimensions ≥ 2 vanish, so we conclude that all the geometric fibers of ρ_Y and thus of ρ_0 are indeed elliptic curves. This complete the proof. $\square$

The following lemma will be helpful.

Lemma 2.2.2. Let $\rho\colon C \to S$ be a separated morphism of schemes with a right inverse $e\colon S \to C$. Then e is a closed immersion.

Proof. By [Sta22] Tag [$01W6$], since ρ is separated and $\rho \circ e = id_S$ is proper, then we may conclude that e is proper. Since e has a left inverse, it is a monomorphism of schemes. Thus by [Sta22], Tag [$04XV$], a proper monomorphism is a closed immersion. $\square$

Observe that this lemma applies immediately to elliptic curves. In particular for an elliptic curve E over an algebraically closed field k, specifying a k-valued point is the same as specifying a closed point in E (since E is of finite type over $Spec(k)$).

Our goal will be to prove a descent property for elliptic curves with respect to fpqc morphisms, and we will reduce it to a descent property of polarized schemes [Ols16], Proposition 4.4.12. We equip each elliptic curve (E, ρ, e) over any S scheme with a particular quasi-coherent sheaf, namely with the ideal sheaf of its section $e: S \to E$. To obtain a polarized scheme, we need to show that this sheaf is a line bundle and that its inverse is ample. Moreover, we will need its construction to be functorial in the sense made more precise below. Before actually dealing with elliptic curves, we will show some more general statements about ideal sheaves.

Definition 2.2.3. Let X be an S-scheme. By an effective Cartier divisor on X/S we mean a closed subscheme C such that:

1. C is flat over S,
2. the ideal sheaf $\mathcal{I}_C \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

Proposition 2.2.4. Let $\rho: C \to S$ be a smooth morphism of relative dimension 1 which is separated and quasi-compact. Let $e: S \to C$ be a section of ρ. The it defines an effective Cartier divisor in C.

Proof. We can apply lemma 2.2.2 to see that it is a closed immersion. In particular it is flat over S. We still need to show that the corresponding ideal sheaf on C is a locally free $\mathcal{O}_C$-module. Firs, this is a Zariski local statement, so we can assume S to be an affine scheme, say $Spec(R)$. Furthermore, we can suppose R to be a noetherian scheme for 2.2.1. In these passages we are also using that being an effective Cartier divisor is stable under base change. The thesis is now obvious through passing to geometric fibers, see Corollary 1.1.5.2 of [KM85]. □

To formalize functoriality properties for the ideal sheaf of the section $e: S \to E$ of an elliptic curve (E, ρ, e) over S we need the following proposition.

Proposition 2.2.5. Let (E, ρ, e) be an elliptic curve over S and (E', ρ', e') an elliptic curve over S'. Let $\mathcal{J} \subset \mathcal{O}_E$ and $\mathcal{J}' \subset \mathcal{O}_{E'}$ be the ideal sheaves of e and e', respectively. Moreover let $(f, g): (E, \rho, e) \to (E', \rho, e')$ be a morphism of elliptic curves. Then:

1. $g^*\mathcal{J}'^{\otimes r} \cong \mathcal{J}^{\otimes r}$ for any $r \in Z$, and these isomorphisms are compatible with composition of morphisms.
2. $\mathcal{J}^{-1}$ is an ample line bundle over S.

Proof. (1) Trivial, since relative effective Cartier divisors behave well under pullbacks (see [KM85], section 1.1.2), and since in the diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "e"] & S' \arrow[d, "e'"] \\ E \arrow[r, "g"] \arrow[d, "\rho"] & E' \arrow[d, "\rho'"] \\ S \arrow[r, "f"] & S' \end{tikzcd} \tag{2.11}$$

the outer and the lower squares are cartesian, hence so is the upper.

(2) To check that $\mathcal{J}^{-1}$ is ample, we use criterion of Corollary 9.6.5 from [GW10], saying that it is enough to check ampleness in every fiber. Moreover, buy [Sta22] Tag [$0D2P$] it suffices to check ampleness in a fpqc cover, hence we can assume $k = \bar{k}$. In this case, the thesis follows from classical results, for example see [Har13] Example 4.3.3.3. Here we identify $\mathcal{J}^{-1}_{Spec(k)}$ with $\mathcal{O}_{Spec(k)}(e_{Spec(k)})$, and one can prove $\mathcal{J}^{-3}_{Spec(k)} \cong \mathcal{O}_{E_{Spec(k)}}(3e_{Spec(k)})$ is very ample. □

After these preliminaries we are now ready to prove the following theorem.

Theorem 2.2.6. $\mathcal{M}_{1,1}$ is a stack in the fpqc topology on (Sch).

Lemma 2.2.7. Let S be a scheme and $\mathcal{F}$ be a category fibered in groupoids over (Sch/S). Suppose that the following conditions are satisfied:

- $\mathcal{F}$ is a stack with respect to the Zariski topology.

- Whenever $V \to U$ is a flat surjective morphism of affine S-scheme, the functor

$$\mathcal{F}(U) \to \mathcal{F}(V \to U)$$

is an equivalence of categories.

Then $\mathcal{F}$ is a stack with respect to the fpqc topology.

Proof. See [Vis04], Lemma 4.25, for a proof. □

In order to prove 2.2.6, We will follow this method:

1. We prove that $\mathcal{M}_{1,1}$ is a prestack in the fpqc topology on (Sch).
2. We prove that $\mathcal{M}_{1,1}$ is a stack in the Zariski topology on (Sch).
3. We prove that $\mathcal{M}_{1,1}$ satisfies the second condition of Lemma 2.2.7.

Proposition 2.2.8. $\mathcal{M}_{1,1}$ is a prestack with respect to the fpqc topology on (Sch).

Proof. We need to show that the functor

$$\mathcal{M}_{1,1}(S) \xrightarrow{\sigma_i^*} \mathcal{M}_{1,1}\ (\{U_i \xrightarrow{\sigma_i} S\}) \tag{2.12}$$

induced by pullbacks is fully faithful for any fpqc covering $\{U_i \xrightarrow{\sigma_i} S\}$. We will check faithfullness first. Let (E, ρ, e) and (E', ρ', e') be two elliptic curves over S and let (id, f), $(id, g)\colon (S, (E, \rho, e)) \to (S, E', \rho', e'))$ be two morphisms in $\mathcal{M}_{1,1}$ which agree on the covering $\{U_i \to S\}$. More precisely, we recall that we have chose pullbacks to define functors $\sigma_i^*\colon \mathcal{M}_{1,1}(S) \to \mathcal{M}_{1,1}(U_i)$ and require $\sigma_i^*(f) = \sigma_i^*(g)$ for all $i \in I$. Observe the map $\{\sigma_i^* E \xrightarrow{\tau_i} E\}$ is an fpqc cover again. The two maps $f, g\colon E \to E'$ are equal since they are equal on U_i and the fpqc site is subcanonical (i. e. the representable presheaves are actually sheaves) after the composition with τ_i.

Next we want to show that our functor is full. Assume that with the same choice of σ_i^*, we are given morphisms

$$((\sigma_i^* E, \sigma_i^* \rho, (id, e \circ \sigma_i)), \alpha^{-1}_{\mathrm{pr}_1, \sigma_i}(E) \circ \alpha_{\mathrm{pr}_2, \sigma_j}(E)) \xrightarrow{\beta_i} ((\sigma_i^* E', \sigma_i^* \rho', (id, e' \circ \sigma_i)), \alpha^{-1}_{\mathrm{pr}_1, \sigma_i}(E') \circ \alpha_{\mathrm{pr}_2, \sigma_j}(E'))$$

compatible with transition maps, where $\alpha_{f,g}(X)\colon f^* g^* X \to (gf)^* X$ is the canonical isomorphism. It is easy to check that the pullback $\mathrm{pr}_1^* \sigma_1^* E$ is canonically isomorphic to $\sigma_i^* E \times_E \sigma_j^* E$, and thus also β_i and β_j coincide when pulled back to the fiber product and composed with the corresponding $\alpha_{i,j}$'s. This implies that (using again that $\{\sigma_i^* E \xrightarrow{\tau_i} E\}$ is an fpqc covering and that E' is a sheaf on the fpqc site) there is a unique map $\beta\colon E \to E'$ so that $\beta \circ \tau_i$ coincides with the composition $\sigma_i^* E_i \xrightarrow{\beta_i} \sigma_i^* E' \xrightarrow{\tau_i'} E'$. So we have a commutative diagram

$$\begin{tikzcd}
\sigma_i^* E \arrow[r, "\beta_i"] \arrow[d, "\tau_i"] & \sigma_i^* E' \arrow[r, "\sigma_i^*(\rho')"] \arrow[d, "\tau_i'"] & U_i \arrow[d, "\sigma_i"] \\
E \arrow[r, "\beta"] & E' \arrow[r, "\rho'"] & S.
\end{tikzcd}$$

For this map to make (id, β) into a morphism of elliptic curves, we need to check $\rho' \circ \beta = \rho$ and $\beta \circ e = e'$. It is enough to check this fpqc-locally.

$$\begin{aligned}
\rho' \beta \tau_i &= \rho' \tau_i' \beta_i && \text{(definition of } \beta) \\
&= \sigma_i \circ \sigma_i^*(\rho') \circ \beta_i && \text{(pullback diagram defining } \sigma_i^* E') \\
&= \sigma_i \circ \sigma_i^*(\rho) && \text{(definition of } \beta_i) \\
&= \rho \circ \tau_i && \text{(pullback diagram defining } \sigma_i^*(E)).
\end{aligned}$$

Similarly, one can check that $\sigma_i^*(\beta) = \beta_i$ using the universal property of pullbacks. Similarly, we have

$$\beta e \sigma_i = \beta \circ \tau_i (id, e \circ \sigma_i) \qquad \text{(definition of } (id, e \circ \sigma_i)$$

$$
\begin{aligned}
&= \tau_i' \circ \beta_i \circ (id, e \circ \sigma_i) && \text{(definition of } \beta) \\
&= \tau_i' \circ (id, e' \circ \sigma_i) && \text{(definition of } \beta_i) \\
&= e' \circ \sigma_i && \text{(definition of } (id, e' \circ \sigma_i)).
\end{aligned}
$$

Altogether, this proves the fullness and thus we have shown that $\mathcal{M}_{1,1}$ is a prestack in the fpqc topology.

□

Proposition 2.2.9. $\mathcal{M}_{1,1}$ is a stack with respect to the Zariski topology.

Proof. We have already shown that $\mathcal{M}_{1,1}$ is a prestack with respect to the fpqc topology, hence the same is true for the Zariski topology. The only thing we need to show is so the "gluing" property for elliptic curves for Zariski coverings.

Let S be any scheme and let $\{U_i \to S\}$ be a Zariski covering of S. We want to prove

$$\mathcal{M}_{1,1}(S) \to \mathcal{M}_{1,1}(\{U_i \to S\})$$

is essentially surjective. Assume that we are given an object with descent data: a family of $(E_i\rho_i, e_i) \in \mathcal{M}_{1,1}(U_i)$ together with isomorphisms of elliptic curves $\varphi_{i,j} \colon \mathrm{pr}_2^* E_j \to \mathrm{pr}_1^* E_i$ satisfying the cocycle condition. We need to construct an elliptic curve E over S with $\sigma_i^* E \cong E_i$ (as elliptic curves). Since morphisms of schemes form a stack in the Zariski topology ([Vis04], section 4.3), we obtain a map $\rho \colon E \to S$ so that $\sigma_i^*(E)$ is isomorphic to E_i and these isomorphisms are compatible with φ_{ij} and with ρ_i and $\sigma_i^*(\rho)$, respectively. It is proper and smooth since these properties can be checked Zariski locally.

For the section, consider the compositions $U_i \xrightarrow{e_i} E_i \cong \sigma_i^* E \to E$. These coincide on intersections, so define a map $e \colon S \to E$ so that $e\sigma_i = e_i$. We need to check $\rho e = id_S$. Again, we can do this Zariski locally, and there it is implied by the commutativity of the following diagram:

$$
\begin{tikzcd}
E \arrow[r, "e_i"] \arrow[drr, equal] & E_i \arrow[r, "\cong"] \arrow[dr, "\rho_i"] & \sigma_i^* E \arrow[r, "\tau_i"] \arrow[d, "\sigma_i^*(\rho)"] & E \arrow[d, "\rho"] \\
 & & U_i \arrow[r, "\sigma_i"] & S.
\end{tikzcd}
$$

Next, we need to check that the pullbacks of (E, e) to geometric fibers are elliptic curves again. Note that since $\{U_i \to S\}$ is a Zariski covering, any map $Spec(\bar{k}) \to S$ factors through some U_i, so that the pullback of E to $Spec(\bar{k})$ is isomorphic to the pullback of E_i to $Spec(\bar{k})$, and also the induced sections coming from e and e_i coincide. Thus, the geometric fibers of ρ are elliptic curves since so are the geometric fibers of ρ_i by assumption.

This complete the proof.

□

The main difficulty now is the fact that "gluing" of schemes given on an fpqc (even an étale) covering does not yield a scheme again. In the étale case, we could in general obtain an algebraic space in this way. So we need some care to see that in the case of elliptic curves (which goes back to the case of polarized schemes), the gluing does work out. Note that this would not work for genus 1 curves without section.

Let $(\mathcal{P}ol)$ the category whose objects are pairs $(f \colon X \to Y, L)$ where f is a proper and flat morphism of schemes and L is a relatively ample invertible sheaf on X. A morphism

$$(f' \colon X' \to Y', L') \to (f \colon X \to Y, L)$$

is a triple (a, b, ε) where

$$a \colon Y' \to Y \quad b \colon X' \to X$$

are morphisms of schemes such that the square

$$
\begin{tikzcd}
X' \arrow[r, "b"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "a"] & Y
\end{tikzcd}
$$

is cartesian and $\varepsilon\colon b^*L \to L'$ is an isomorphism of invertible sheaves on X'.

There is a functor

$$\begin{aligned}(\mathcal{P}ol) &\to (\mathit{Schemes})\\ (f\colon X \to Y, L) &\mapsto Y\end{aligned}$$

which makes $(\mathcal{P}ol)$ into a fibered category over the category of schemes.

Proposition 2.2.10. $(\mathcal{P}ol)$ is a stack in the fpqc topology.

Proof. See [Ols16], Proposition 4.4.12. □

This result will be helpful for the conclusion of the proof of Theorem 2.2.6.

Lemma 2.2.11. Whenever $V \to U$ is a flat surjective morphism of affine S-scheme, the functor

$$\mathcal{M}_{1,1}(U) \to \mathcal{M}_{1,1}(V \to U)$$

is an equivalence of categories.

Proof. We only need to check that it is essentially surjective. Let an elliptic curve $\rho\colon E \to V$ with section $e\colon V \to E$ be given, together with descent data, i. e. an isomorphism $\beta\colon pr_2^*E \to pr_1^*E$ of elliptic curves over $V \times_U V$ satisfying the appropriate cocycle condition. Observe that Proposition 2.2.5 implies that we have a morphism of fibered categories

$$\begin{aligned}\mathcal{M}_{1,1} &\to (\mathcal{P}ol)\\ (S', (E', \rho', e')) &\mapsto (\rho'\colon E' \to S', \mathcal{J}'^{-1})\end{aligned}$$

where $\mathcal{J}'^{-1}$ is the ideal sheaf corresponding to e'. This implies in particular that we have a 2-commutative diagram of categories

$$\begin{tikzcd} \mathcal{M}_{1,1}(U) \arrow[r] \arrow[d] & \mathcal{P}ol(U) \arrow[d] \\ \mathcal{M}_{1,1}(V \to U) \arrow[r] & \mathcal{P}ol(V \to U). \end{tikzcd}$$

Because of the Proposition 2.2.10, we conclude there is a scheme X and a flat proper morphism of finite presentation $q\colon X \to U$ and an isomorphism $\sigma\colon \varphi^*X \to E$ of schemes over V, satisfying certain compatibilities. Actually we also get a relatively very ample invertible sheaf $\mathcal{L}$ on X and an isomorphism $\sigma^*\pi^*\mathcal{L} \cong \mathcal{J}^{-1}$ of $\mathcal{O}_E$-modules, where π denotes the projection $\varphi^*X \to X$, but we will not use this. In particular, this means that σ is an isomorphism of descent data, and more explicitly, we have a commutative diagram

$$\begin{tikzcd} pr_2^*\varphi^*X \arrow[r, "pr_2^*\sigma"] \arrow[d, "\gamma"] & pr_2^*E \arrow[d, "\beta"] \\ pr_1^*\varphi^*X \arrow[r, "pr_1^*\sigma"] & pr_1^*E. \end{tikzcd} \tag{2.13}$$

The last piece of data we need to construct is a section for q. To do so, we will use that X is a sheaf in the fpqc topology. Observe that we have a map $V \to X$ defined as

$$V \xrightarrow{e} E \xrightarrow{\sigma^1} \varphi^*X \xrightarrow{\pi} X.$$

We have to show that precomposing this map with either projection $pr_i\colon V \times_U V \to V$ yields the same result. Since β is an isomorphism of elliptic curves, it satisfies $\beta \circ pr_2^*(e) = pr_1^*(e)$. Thus we have

$$\begin{aligned}
\pi \circ \sigma^{-1} \circ e \circ pr_1 &= \pi \circ \sigma^{-1} \circ pr_1 \circ pr_1^*(e) && \text{(as in (2.11))}\\
&= \pi \circ \sigma^{-1} \circ pr_1 \circ \beta \circ pr_2^*(e) && \text{(definition of } \beta)\\
&= \pi \circ pr_1 \circ pr_1^*(\sigma)^{-1} \circ \beta \circ pr_2^*(e) && \text{(definition of } pr_1^*(\sigma))\\
&= \pi \circ pr_1 \circ \gamma \circ pr_2^*(\sigma)^{-1} \circ pr_2^*(e) && \text{(commutativity of (2.13))}
\end{aligned}$$

$$\begin{aligned} &= \pi \circ \sigma^{-1} \circ pr_2 \circ pr_2^*(e) && \text{(definition of } pr_2^*(\sigma)) \\ &= \pi \circ \sigma^{-1} \circ e \circ pr_2 && \text{(as in (2.11)).} \end{aligned}$$

Thus we obtain a map $j\colon U \to X$ so that $j \circ \varphi = \pi \circ \sigma^{-1} \circ e$. We still need to show that $qj = id_U$. Since U is a sheaf in the fpqc topology, we only need to check $qj\varphi = \varphi$ which is straightforward. Next we conclude by [Gro66], Corollary 17.7.3, that $q\colon X \to U$ is actually smooth since its pullback along the faithfully flat and quasi-compact map φ is smooth.

We are left to show that for any algebraically closed field k and any morphism $Spec(k) \to U$, the pullback (X_k, j_k) of (X, j) is an elliptic curve over k. Recall that $U = Spec(A)$ and $V = Spec(B)$ were assumed to be affine. Then we know that $Spec(k) \times_U V \cong Spec(B \otimes_A k)$. Note that $B \otimes_A k$ is not the 0-ring since the map $A \to B$ was assumed to be faithfully flat, thus it has some maximal ideal $\mathfrak{m}$. Let L be the algebraic closure of the quotient $B \otimes_A k/\mathfrak{m}$; then we have the commutative diagram

$$\begin{tikzcd} Spec(L) \arrow[r] \arrow[d] & V \arrow[d] \\ Spec(k) \arrow[r] & U. \end{tikzcd}$$

Let E_L be the pullback of E along $Spec(L) \to V$. Note than the diagram

$$\begin{tikzcd} E_L \arrow[r, "\rho_L"] \arrow[d, "g"] & Spec(L) \arrow[d, "\tilde{g}"] \\ X_k \arrow[r, "q_k"] & Spec(k) \end{tikzcd}$$

is cartesian. It is easy to check that also the pullback of the section $j_k\colon Spec(k) \to X_k$ is the section e_L, and recall we assumed that (E_L, e_L) is an elliptic curve over L. Thus we conclude by Lemma **??** that (X_k, j_k) is an elliptic curve over k.

This finishes the proof that the triple (X, q, j) constructed above is an elliptic curve over U whose pullback is up to isomorphism given by the descent data we started with. □

This concludes the proof that $\mathcal{M}_{1,1}$ is indeed a stack with respect to the fpqc topology.

2.3. Weierstrass equation for stack

The main result we prove in this section is that any elliptic curve is Zariski locally given by a Weierstrass equation.

Lemma 2.3.1. Let R be an arbitrary ring, and let $f_1, \ldots, f_n$ be homogeneous polynomials in $R[T_1, \ldots, T_k]$ with $n \leq k-1$. Then

$$X = Proj(R[T_1, \ldots, T_k]/(f_1, \ldots, f_n)) \tag{2.14}$$

is smooth over Spec(R) of relative dimension $k-1-n \iff$ for any ring homomorphism into a field $\alpha\colon R \to k$, the scheme

$$X_k = Proj(k[T_1, \ldots, T_k]/(\alpha(f_1), \ldots, \alpha(f_n)) \tag{2.15}$$

is smooth over k of relative dimension $k-1-n$.

Proof. $\Longrightarrow$ If $X \to Spec(R)$ is smooth of relative dimension $k-1-n$, so is any pullback $X \times_{Spec(R)} Spec(k) \to Spec(k)$.

$\Longleftarrow$ For the converse, it is enough to show that $X \to Spec(R)$ is smooth of relative dimension $k-1-n$ when restricted to each $D_+(T_i)$. Thus, we need to check that the map

$$R \to R[T_1, \ldots, \bar{T}_i, \ldots, T_k]/(f_1(T_i = 1), \ldots, f_n(T_i = 1))$$

is smooth of relative dimension $k-1-n$. We already know that the restriction to $D_+(T_i) \subset X_k$ is smooth of the same relative dimension for any $\alpha\colon R \to k$, so that

$$k \to k[T_1, \ldots, \bar{T}_i, \ldots, T_k]/(\alpha(f_1(T_i = 1)), \ldots, \alpha(f_n(T_i = 1)))$$

is smooth of the same dimension, and also standard smooth. This means that the matrix

$$(\frac{\partial \alpha(f_r(T_i=1))}{\partial T_s})_{1\leq r\leq n,\ 1\leq s\leq k,\ s\neq i}$$

has rank n. By Definition 6.14 of [GW10], this corresponds exactly to the smoothness of $X \to Spec(R)$. ☐

For the next lemma we use the same notation of section 2.1.

Corollary 2.3.2. Let R be any ring. Then the closed subscheme of $\mathbb{P}^2_R$ cut out by the Weierstrass equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3. \tag{2.16}$$

is an elliptic curve over R if Δ is invertible in R.

Moreover, if this equation cuts out a smooth scheme over $Spec(R)$, then Δ is invertible in R.

Proof. Denote with E the scheme $Proj(R[X,Y,Z]/(Y^2Z+a_1XYZ+a_3YZ^2 = X^3+a_2X^2Z+a_4XZ^2+a_6Z^3)$. This comes with a natural map $E \to Spec(R)$. This map is proper since it is the composition of a closed immersion $E \to \mathbb{P}^2_R$ and the map $\mathbb{P}^2_R \to Spec(R)$ (both proper for known facts). We use the Lemma 2.3.1 to reduce to study the case $R = k$. Note that it is enough to consider algebraically closed fields. Hence the result follows from Proposition 2.1.2. ☐

We need the following technical lemma.

Lemma 2.3.3. Let $f\colon X \to Y$ be a morphism of proper S-schemes such that for every geometric point $Spec(k) \to S$, the pullback $f_k\colon X_k \to Y_k$ is a closed immersion. Then f is a closed immersion.

Proof. First, by [Sta22] Tag [$01W6$] we have that f itself is proper. Now we exploit Corollary 18.12.6 of [Gro67] combined with Proposition in [Sta22] Tag [$01S4$]. Therefore, being f a closed immersion is equivalent to the fact that for every $y \in Y$ with residue field $k(y)$, the pullback map

$$X_{k(y)} := X \times_Y Spec(k(y)) \to Spec(k(y))$$

is universally injective. Let $s = h(y)$, where $h\colon Y \to S$. The map $Y \to S$ induces a map of residue fields $k(s) \to k(y)$ and a commutative diagram

$$\begin{tikzcd} Spec(k(y)) \arrow[r] \arrow[d] & Y \arrow[d] \\ Spec(k(s)) \arrow[r] & S, \end{tikzcd}$$

so that we also obtain a map $Spec(k(y)) \to Y_{k(s)} := Y \times_S Spec(k(s))$ compatible with the above morphisms. In particular, the map $Spec(k(y)) \to Y$ factors through $Y_{k(s)} \to Y$, and the squares below are pullback squares:

$$\begin{tikzcd} X_{k(y)} \arrow[r] \arrow[d] & X_{k(s)} \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(k(y)) \arrow[r] & Y_{k(s)} \arrow[r] & Y. \end{tikzcd} \tag{2.17}$$

Moreover, observe that $k(y)$ is indeed the residue field $k(y')$ of its image point y' in $Y_{k(s)}$, since the morphism $Y_{k(s)} \to Y$ by construction has the property $k(y) \subset k(y')$. If we prove that $X_{k(s)} \to Y_{k(s)}$ is a closed immersion, then using Corollary 18.12.6 [Gro67] again, the left vertical arrow of the diagram

$$\begin{tikzcd} (X_{k(s)})_{k(y')} = X_{k(y)} \arrow[r] \arrow[d] & X_{k(s)} \arrow[d] \\ Spec(k(y')) = Spec(k(y)) \arrow[r] & Y_{k(s)} \end{tikzcd}$$

is universally injective, which is exactly what we need to conclude.

Now it remains to prove that $X_{k(s)} \to Y_{k(s)}$ is a closed immersion. We already know by assumption that $X_{\bar{k(s)}} \to Y_{\bar{k(s)}}$ is a closed immersion. Since $Spec(\bar{k(s)}) \to Spec(k(s))$ is fpqc, so is $Y_{\bar{k(s)}} \to Y_{k(s)}$, and thus by fpqc descent (see [Vis04] Proposition 1.15) the map $X_{k(s)} \to Y_{k(s)}$ is a closed immersion. This complete the proof. ☐

Our next objective is to provide local Weierstrass equations for all elliptic curves.

Theorem 2.3.4. Zariski locally, any elliptic curve is given by a Weierstrass form.

Proof. By 2.2.1, it is enough to show that any elliptic curve $(E, \rho\colon E \to Spec(R), e)$ for a noetherian ring R is Zariski locally cut out by a Weierstrass equation. Recall that we have shown in Proposition 2.2.5 that the ideal sheaf $\mathcal{I}$ of the closed immersion $e\colon Spec(R) \to E$ is an invertible line bundle, and also that the formation of $\mathcal{L} := \mathcal{I}^{-1}$ is compatible with base change. By definition, we have an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_E \longrightarrow e_*\mathcal{O}_{Spec(R)} \longrightarrow 0, \tag{2.18}$$

which yields for any $n \geq 0$ the short exact sequence

$$0 \longrightarrow \mathcal{L}^{\otimes n} \longrightarrow \mathcal{L}^{\otimes n+1} \longrightarrow e_*\mathcal{O}_{Spec(R)} \otimes \mathcal{L}^{\otimes n+1} \longrightarrow 0. \tag{2.19}$$

Next, we want to show that $\rho_*(\mathcal{L}^{\otimes n})$ is a locally free module of rank n for $n \geq 1$. We will use again the variant of cohomology and base change of [Vis04], Proposition 4.37. To apply it, we need to compute $H^1(E_k, \mathcal{L}_k^{\otimes n})$, and it is enough to do so for algebraically closed field $\bar{k} = k$. Using Serre duality and the fact that the dualizing sheaf of an elliptic curve over an algebraically closed field is trivial, we conclude

$$H^1(E_k, \mathcal{L}_k^{\otimes n}) \cong H^0(E_k, \mathcal{L}_k^{\otimes -n}).$$

We can identify $\mathcal{L}_k \cong \mathcal{O}_{E_k}(e_{Spec(k)})$. We conclude that negative powers of $\mathcal{L}_k$ do not have non-trivial global sections. Hence, we may apply [Vis04], Proposition 4.37 to conclude that $\rho_*(\mathcal{L}^{\otimes n})$ is a locally free module and its formation commutes with base change. To determine the rank, we use again the pullback to an algebraically closed field k. Riemann-Roch's theorem immediately implies that the rank is n and so is the rank of $\rho_*(\mathcal{L}^{\otimes n})$.

Next, we want to show that the quotient of the pushforward map $\rho_*(\mathcal{L}^{\otimes n}) \to \rho_*(\mathcal{L}^{\otimes (n+1)})$ is a locally free module of rank 1. We can choose an affine covering of $Spec(R)$ trivializing both of these locally free sheaves. For any $Spec(A)$ in this covering, the quotient corresponds (via global sections) to an A-module M in the exact sequence of the form

$$0 \longrightarrow A^n \longrightarrow A^{n+1} \longrightarrow M \longrightarrow 0 \tag{2.20}$$

Given any A-algebra A', we pull back to $Spec(A')$ and obtain an elliptic curve (E', ρ', e'). We know that the inverse $\mathcal{L}'$ of the ideal sheaf of e' is exactly the pullback of $\mathcal{L}$ (and the same for all its powers). Since the formation of $\rho_*(\mathcal{L}^{\otimes n})$ commutes with base change, we conclude that the resulting map $A^n \otimes_A A' \to A^{n+1} \otimes_A A'$ is injective, so that $Tor^1_A(M, A') = 0$. This is in particular true for all square 0 extensions $A' = A \oplus N$, where we mean that given an A-module N, we give to $A \oplus N$ a structure of A-algebra with multiplication $(a_1, n_1) \cdot (a_2, n_2) = (a_1a_2, a_1n_2 + a_2n_1)$. Since $Tor^1_A(M, A') = Tor^1_A(M, A) \oplus Tor^1_A(M, N)$ we conclude $Tor^1_A(M, N) = 0$ for every A-module, i.e. M is flat. Since R and thus A are noetherian, and M is flat and finitely generated, we can conclude that M is locally free, and then it is necessarily of rank 1.

Following similar arguments, and by possibly restricting further, we may assume $\rho_*(\mathcal{L}^{\otimes n})$ for $n \in \{1, 2, 3, 6\}$ and also the quotients $\rho_*(\mathcal{L}^{\otimes 2})/\rho_*(\mathcal{L}^{\otimes 1})$ and $\rho_*(\mathcal{L}^{\otimes 3})/\rho_*(\mathcal{L}^{\otimes 2})$ to be trivial over $Spec(A)$. Moreover, we will use the product structure on $\oplus_{n\geq 0}\mathcal{L}^{\otimes n}$.

Fixing identifications, we may choose the basis 1 for

$$A \cong \Gamma(Spec(A), \rho_*\mathcal{L}) \cong \Gamma(E \times_{Spec(R)} Spec(A), \mathrm{pr}_{2*}\mathrm{pr}_1^*\mathcal{L}).$$

Using freeness of the quotients, we can also choose bases $1, x$ and $1, x, y$ for the cases $n = 2$ and $n = 3$ respectively. This defines the elements

$$1, x, x^2, x^3, y, xy, y^2 \in \Gamma(Spec(A), \rho_*(\mathcal{L}^{\otimes 6})) \cong A^6. \tag{2.21}$$

Now we want to prove that there exist $a_1, \ldots, a_7 \in A$ such that

$$a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 = 0, \tag{2.22}$$

with $a_6a_7 \neq 0$. Equivalently, it suffices to prove the map $A^7 \to \Gamma(Spec(A), \rho_*(\mathcal{L}^{\otimes 6})) \cong A^6$ is surjective. Being surjective is a local property, and we also can verify this only pullbacking for maximal ideals of A, so the thesis follows from the Theorem 2.1.4, part (1). Moreover, the arrow $E_A := E \times_{Spec(R)} Spec(A) \to \mathbb{P}^2_A$ using the map given by the sections $x, y, 1$, is a closed immersion, since for lemma 2.3.3 it suffices to check the pullback to each geometric point, and for these the thesis follows again from the Theorem 2.1.4, part (1).

One can check now that the map defined by sections $x, y, 1$ factors through the closed subscheme $V_+(F) = Proj(A[X,Y,Z]/(F))$ of $\mathbb{P}^2_A$, where F is the homogeneous variant of 2.22. Our goal now is to show that the resulting map $E_A \to V_+(F)$ is an isomorphism of schemes over $Spec(A)$. To do so, we need to show that the corresponding ideal sheaf $\mathcal{J}$ vanishes; recall it is defined as the kernel of

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{V_+(F)} \longrightarrow j_*\mathcal{O}_{E_A} \longrightarrow 0. \tag{2.23}$$

Once again we exploit that, if we base-change the whole situation to an algebraically closed field k, the pullback $\mathcal{J}_k$ of $\mathcal{J}$ to $V_+(F_k)$ vanishes. Indeed, if we pullback the exact sequence 2.23 along a morphism $\mathbb{P}^2_k \to \mathbb{P}^2_A$, the sequence remains exact since $j_*\mathcal{O}_{E_A}$ is flat over $Spec(A)$. Moreover we can use cohomology and base change for affine maps to conclude that the pulled back exact sequence is of the form

$$0 \longrightarrow \mathcal{J}_k \longrightarrow \mathcal{O}_{V_+(F_k)} \longrightarrow (j_k)_*\mathcal{O}_{E_k} \longrightarrow 0. \tag{2.24}$$

Here again we use the Theorem 2.1.4 to say $\mathcal{J}_k = 0$. So, if we restrict the original exact sequence 2.23 to standard opens in $\mathbb{P}^2_A$, we in each case get a corresponding sequence of A-module

$$0 \longrightarrow J \longrightarrow S \longrightarrow N \longrightarrow 0, \tag{2.25}$$

where we in addition know that S is a finitely generated A-algebra, thus in particular a noetherian ring. Moreover, after tensoring with k for any ring homomorphism $A \to k$ to an algebraically closed field k, we have deduced $J \otimes k = 0$. Then also $J \otimes k = 0$ holds for any k field. Now given a ring homomorphism $S \to k$, we obtain by composition with the morphism as algebra $A \to S$, a map $A \to k$. In particular, the tensor product $J \otimes_S k$ is a quotient of $J \otimes_A k$ and thus 0 itself. This applies in particular to $k = S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset S$. Since S is noetherian, J is finitely generated as S-module, so we can apply Nakayama's Lemma to conclude $J = 0$. Notice that this isomorphism is an isomorphism of elliptic curves, since it respects the sections. In fact, by construction the section $e_A \colon Spec(A) \to E_A \cong V_+(F)$ factors through the section of $V_+(F)$ given by "$[0,1,0]$". With the same argument used before, it is an isomorphism since it can be verified on each fiber by Theorem 2.1.4 and after one can lift the argument using the noetherian hypothesis and Nakayama's Lemma. This concludes the proof.

□

An important consequence of this Theorem is the fact that $\mathcal{M}_{1,1}$ is a quotient stack. In fact, let U be the scheme

$$U := Spec(\mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \frac{1}{\Delta}]), \tag{2.26}$$

Let G denote the group scheme with underlying scheme $Spec(\mathbb{Z}[u^{\pm}, r, s, t])$ with group law defined by

$$(u', r', s', t') \cdot (u, r, s, t) = (uu', u^2r' + r, us' + s, u^3t' + u^2r's + t). \tag{2.27}$$

The identity is the element $(1, 0, 0, 0)$. This group acts on U via

$$\begin{aligned}(u,r,s,t)\cdot(a_1,a_2,a_3,a_4,a_6) = (&\frac{a_1+2s}{u}, \frac{a_2 - sa_1 + 3r - s^2}{u^2},\\ &\frac{a_3 + ra_1 + 2t}{u^3}, \frac{a_4 - sa_3 + 2ra_2 - (t+rs)a_1 + 3r^2 - 2st}{u^4},\\ &\frac{a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1}{u^6}).\end{aligned}$$

One can check that it is well defined since $(u,r,s,t)\cdot\Delta = u^{12}\Delta$ and it is indeed an action. Moreover, given an elliptic curve of the form

$$Spec(\mathcal{O}(U)[x,y]/(y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6)),$$

the action on the coefficients induces an action on it, providing a new elliptic curve. In fact, if we denote with (x', y') the new coordinates, one can easily observe that this action is exactly:

$$(X', Y') \mapsto \begin{cases} X = u^2 X' + r, \\ Y = u^3 Y' + sX' + t, \end{cases} \tag{2.28}$$

and these are exactly the isomorphisms between elliptic curves guaranteed by Theorem 2.1.4 part (2).

Theorem 2.3.5. The morphism $U \to \mathcal{M}_{1,1}$ defined by the so called tautological elliptic curve $\mathcal{E} := \mathrm{Proj}(\mathbb{Z}[x, y, z]/(Y^2 Z + a_1 XYX + a_3 YZ^2 = X^3 + a_2 X^2 Z + a_4 XZ^2 + a_6 Z^3)$ is smooth and surjective.

Moreover, we have the stack isomorphism $\mathcal{M}_{1,1} \cong [U/G]$. In particular $\mathcal{M}_{1,1}$ is an algebraic stack with respect to the fpqc topology.

Proof. Observe that from Theorem 2.3.4 we have that each elliptic curve $(E, \rho \colon E \to S, e)$ is Zariski locally given by a Weierstrass equation. Hence we have locally on an étale cover $\{S_i \to S\}$ a morphism $f_i \colon S_i \to U$ such that $f_i^* \mathcal{E} = E_i$. On the intersections these morphisms glue through the groups $G_{S_{ij}}$ and this give to $U \times_{\mathcal{M}_{1,1}} S \to S$ a structure of G_S-bundle. Moreover, we observe that the map $U \to \mathcal{M}_{1,1}$ is G-invariant. From Theorem 1.7.8 it follows that $\mathcal{M}_{1,1} \cong [U/G]$. □

Notice that there are several presentation of the stack $\mathcal{M}_{1,1}$ and its pull-backs. One may obtain these generalizing the Weierstrass equations like in the classical setting, exploiting Theorem 2.3.4 and after applying again Theorem 1.7.8 as in the previous proof. We see now some examples.

Example 2.3.6. There is the *universal Legendre family* E_2 over the ring $R_2 := \mathbb{Z}[\frac{1}{2}, \lambda][\frac{1}{\lambda(1-\lambda)}]$, defined as

$$E_2 := Proj(R_2[x, y, z]/(y^2 z - x(x - z)(x - \lambda z)),$$

with the chosen $\Omega_2 := -\frac{dx}{2y}$ and two specified 2-torsion points P_2, Q_2 such that $x(P_2) = 0$ and $x(Q_2) = 1$.

The quadruple $(E_2, \Omega_2, P_2, Q_2)$ is universal, in the sense that given any quadruple $(E/S, \omega, P_2', Q_2')$, where S is a $\mathbb{Z}[1/2]$-scheme, there exists a unique cartesian diagram

$$\begin{array}{ccc} E & \longrightarrow & E_2 \\ \downarrow & & \downarrow \\ S & \longrightarrow & Spec(R_2) \end{array}$$

such that the pull-back of Ω_2 is ω and the pull-backs of the effective Cartier divisors P_2 and Q_2 are P_2' and Q_2' respectively.

Example 2.3.7. There is the *universal family* E_3 *of naive level 3 structure* over the ring $R_3 := \mathbb{Z}[\frac{1}{3}, b, c][\frac{1}{\Delta(b,c)}]/(c^3 + 3bc^2 + 3b^2 c)$, defined as

$$E_3 := Proj(R_3[x, y, z]/(y^2 z + (3c - 1)xyz - (3c^2 + 3bc + b)yz^2 - x^3),$$

with $\Delta(b, c)$ which represents the discriminant in the coordinates b and c. Moreover, this is equipped with the data of the chosen $\Omega_3 := -\frac{dx}{2y}$ and two specified 3-torsion points $P_3 = (0, 0)$ and $Q_3 = (c, b+c)$.

The quadruple $(E_3, \Omega_3, P_3, Q_3)$ is universal, in the sense that given any quadruple $(E/S, \omega, P_3', Q_3')$, where S is a $\mathbb{Z}[1/3]$-scheme, there exists a unique cartesian diagram

$$\begin{array}{ccc} E & \longrightarrow & E_3 \\ \downarrow & & \downarrow \\ S & \longrightarrow & Spec(R_3) \end{array}$$

such that the pull-back of Ω_3 is ω and the pull-backs of the effective Cartier divisors P_3 and Q_3 are P_3' and Q_3' respectively.

In Chapter 3, in the proof of Proposition 3.6.1 we will use a variation of this given by the Hesse presentation.

Since these two examples are standard facts, for the sake of brevity, we omit the details and invite the reader to consult [KM85], Example 2.2.8 and 2.2.10 respectively.

Consider the morphisms $Spec(R_2) \to \mathcal{M}_{1,1}$ and $Spec(R_3) \to \mathcal{M}_{1,1}$ defined by the respective universal families. They induce

$$Spec(R_2) \to \mathcal{M}_{1,1} \times Spec(\mathbb{Z}[1/2])$$

and

$$Spec(R_3) \to \mathcal{M}_{1,1} \times Spec(\mathbb{Z}[1/3])$$

which are both surjective. Moreover, they are both étale (for a complete proof see [KM85], Theorem 2.3.1), and so the map

$$Spec(R_2) \bigsqcup Spec(R_3) \to \mathcal{M}_{1,1}$$

is an étale atlas. We have proved

Proposition 2.3.8. The stack $\mathcal{M}_{1,1}$ is Deligne-Mumford.

We conclude the section with a variation of the discussion done before of Theorem 2.3.5.

Let W be the scheme

$$W := Spec(\mathbb{Z}[a_1, a_2, a_3, a_4, a_6]) - V(\Delta, c_4).$$

The difference with the scheme U defined in 2.26, is that Δ is allowed to be zero, but in that case $c_4 \neq 0$. Let G denote the group scheme defined above (2.27), with the action extended in the natural way to W.

Definition 2.3.9. We denote with $\bar{\mathcal{M}}_{1,1}$ the quotient stack $[W/G]$. It is the *standard compactification of the stack* $\mathcal{M}_{1,1}$. We denote with $\bar{\mathcal{M}}_{1,1,S}$ the pull back $S \times_{Spec(\mathbb{Z})} \bar{\mathcal{M}}_{1,1}$ for a scheme S.

Observe that in light of Proposition 2.1.2, what we have just defined is the same as the stack $\mathcal{M}_{1,1,S}$ to which the nodal curves have been added.

In fact, one could proceed in a different way to define the stack $\bar{\mathcal{M}}_{1,1}$. First of all, one define the *generalized elliptic curves* over a scheme S as the datum of a proper flat finitely presented morphisms $E \to S$ each of whose geometric fibers is either a smooth connected curve of genus 1 or a nodal curve, together with a smooth section $e \colon S \to E$. After having defined morphisms of generalized elliptic curves, in a manner analogous to the smooth case, we can consider $\bar{\mathcal{M}}_{1,1}$ as the category fibered over the category of schemes, such that $\bar{\mathcal{M}}_{1,1}(T)$ is the groupoid of generalized elliptic curves, for every scheme T. Then, one can prove that this is a stack, following the same argumentation for $\mathcal{M}_{1,1}$, with slight changes. Finally, after having proved a result similar to Theorem 2.3.4 which includes nodal curves too, one can prove that the stack obtained is isomorphic to $[W/G]$.

By the way, to make the exposition brief and not to be redundant, we have followed a direct approach simply saying that $\bar{\mathcal{M}}_{1,1}$ is $[W/G]$. For more details about this different approach see for [DR73] on Chapter 4.

2.4. The Automorphism Group

As we have already explained, it is essential to know the structure of the automorphisms of the object of a moduli problem.

In the case of elliptic curves, starting from a Weierstrass equation, it is easy to compute the automorphism group.

Here we present the proof of the structure of the automorphism group, entirely taken by [Sil09], Theorem 10.1, since we need some of the details of this proof in the next chapter.

Theorem 2.4.1. Let E/k be an elliptic curve over an algebraically closed field. Then its automorphism group $Aut(E)$ is given by the following table:

#Aut(E)	**j(E)**	**char(k)**
2	$j(E) \neq 0$, 1728	any
4	$j(E) = 1728$	$char(k) \neq 2\,3$
6	$j(E) = 0$	$char(k) \neq 2\,3$
12	$j(E) = 0 = 1728$	$char(k) = 3$
24	$j(E) = 0 = 1728$	$char(k) = 2$

Proof. ($\mathbf{Char(k) \neq 2, 3}$). We know that E is given by an equation

$$E : y^2 = x^3 + Ax + B,$$

and every automorphism of E has the form

$$\begin{aligned} x &= u^2 x' \\ y &= u^3 y', \end{aligned}$$

for such a $u \in k^*$. Such a substitution gives us

$$y'^2 = x'^3 + \frac{A}{u^4} x' + \frac{B}{u^6}.$$

Therefore, it is an isomorphism if and only if

$$\begin{aligned} \frac{A}{u^4} &= A, \\ \frac{B}{u^6} &= B. \end{aligned}$$

If $AB \neq 0$, i. e. if $j(E) \neq 0$, 1728, then the only possibilities are $u = \pm 1$. Similarly, if $B = 0$, then $j(E) = 1728$ and $u^4 = 1$, and if $A = 0$, then $j(E) = 0$ and $u^6 = 1$. Hence $Aut(E)$ is cyclic of order 2, 4 *or* 6, depending on whether $AB \neq 0$, $B = 0$, $A = 0$.

($\mathbf{Char(k) = 3}$ **and** $\mathbf{j(E) \neq 0}$). In this case E has a Weierstrass equation of the form

$$y^2 = x^3 + a_2 x^2 + a_6.$$

The only substitutions preserving this type of equation are

$$\begin{aligned} x &= u^2 x' \\ y &= u^3 y'. \end{aligned}$$

An easy computation shows that it represents an automorphism if and only if $a_2' u^2 = a_2$, i.e. if and only if $u^2 = 1$, so $Aut(E) \cong \mathbb{Z}/2\mathbb{Z}$.

($\mathbf{Char(k) = 3}$ **and** $\mathbf{j(E) = 0}$). The curve has a Weierstrass equation of the form

$$y^2 = x^3 + a_4 x + a_6.$$

The substitutions preserving this form look like

$$\begin{aligned} x &= u^2 x' + r \\ y &= u^3 y'. \end{aligned}$$

Note that we have that a_4, $a_4' \neq 0$. An automorphism is obtained by choosing u and r to satisfy $a_4' = a_4$ and $a_6' = a_6$. However

$$\begin{aligned} a_4 u^4 &= a_4', \\ r^3 + a_4 r + a_6 - u^6 a_6' &= 0, \end{aligned}$$

hence u and r satisfy

$$\begin{aligned} u^4 &= 1 \\ r^3 + a_4 r + (1 - u^2) a_6 &= 0. \end{aligned}$$

Since $a_4 \neq 0$ there are exactly 12 such pairs (u, r) making up $Aut(E)$. In particular, by the classification of groups of order 12, it is easy to see that the automorphism group is isomorphic to $\mathbb{Z}/4\mathbb{Z} \ltimes \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \ltimes S_3$.

($\mathbf{Char(k) = 2}$ **and** $\mathbf{j(E) \neq 0}$). E is given by equation of the form

$$y^2 + xy = x^3 + a_2x^2 + a_6,$$

with $a_6 \neq 0$ since $j(E) \neq 0$. The substitutions preserving this form look like

$$\begin{aligned} x &= x' \\ y &= y' + sx. \end{aligned}$$

So automorphisms come from taking s to be a root of the equation

$$s^2 + s = 0,$$

hence $s \in \{0, 1\}$.

($\mathbf{Char(k) = 2}$ **and** $\mathbf{j(E) = 0}$). Here E is given by equations of the form

$$y^2 + a_3y = x^3 + a_4x + a_6,$$

and allowable substitutions look like

$$\begin{aligned} x &= u^2x' + s^2 \\ y &= u^3y' + su^2x' + t. \end{aligned}$$

By assumption, $a_3 \neq 0$. To make an automorphism we must choose u, s, t to satisfy the equations

$$\begin{aligned} u^3 &= 1, \\ s^4 + a_3s + (1-u)a_4 &= 0, \\ t^2 + a_3t + s^6 + a_4s^2 &= 0. \end{aligned}$$

Since $a_3 \neq 0$, we see that $Aut(E)$ has order 24.

□

The reason why an entire chapter has been devoted to the group of automorphisms of elliptic curves is that the knowledge of the automorphisms of the points of the stack is fundamental to deduce some of its geometric properties. Actually, in chapter 3, the automorphisms will be crucial to determine the structure of the Picard group of $\mathcal{M}_{1,1,S}$.

Finally, observe that it is in fact the presence of the automorphism of the curves to obstruct the construction of a fine moduli space (i. e. the stack $\mathcal{M}_{1,1}$ is not representable).

2.5. The coarse moduli space of $\mathcal{M}_{1,1}$

Define the j-invariant for an elliptic curve over S defined by a Weierstrass equation at the same way of the classical case:

$$j_{(E,e)} = (b_2^2 - 24b_4)^3/\Delta \in \Gamma(S, \mathcal{O}_S). \tag{2.29}$$

Proposition 2.5.1. Let S be a scheme, and $f_1, f_2 \colon S \to U$ two morphisms defining two elliptic curves (E, e) and (E', e') over S. Let j (respectively j') be the j-invariant defined using elements $f_1^\sharp(\underline{a})$ (respectively $f_2^\sharp(\underline{a})$). If (E, e) and (E', e') are isomorphic elliptic curves, then $j = j'$.

Proof. When S is a field, we reduce to the case in which it is algebraically closed and this is exactly the Theorem 2.1.2, part (2). When S is a domain is simple, because the equality $j = j'$ can be verified at the generic point (i. e. we are tensoring for $\mathbb{Q}(\Gamma(S, \mathcal{O}_S))$ the fraction field). Similarly, with the same approach we prove the thesis when S is reduced, since we can verify the equality at each generic point.

Finally, observe how the general case can be reduced to the case of S reduced as follows. Consider the fiber product

$$V := U \times_{\mathcal{M}_{1,1}} U$$

Let $\mathcal{E} \to U$ be the tautological elliptic curve, and let $\varphi\colon \mathrm{pr}_1^*\mathcal{E} \to \mathrm{pr}_2^*\mathcal{E}$ over V. If there is an isomorphism $\lambda\colon (E,e) \to (E',e')$, then the element $((E,e),(E',e'),\lambda) \in V$. Hence giving such an isomorphism is equivalent to give a morphism $\gamma\colon S \to V$ such that $\mathrm{pr}_1 \circ \gamma = f_1$ and $\mathrm{pr}_2 \circ \gamma = f_2$. It therefore suffices to verify that the j-invariant of $\mathrm{pr}_1^*\mathcal{E}$ and $\mathrm{pr}_2^*\mathcal{E}$ over V are equal. Now since U is a reduced scheme, and $\mathrm{pr}_1,\, \mathrm{pr}_2\colon V \to U$ are smooth, the scheme V is also reduced, and so the general case follows from the reduced one.

□

Now, given an elliptic curve $(E,e)/S$ defined over a scheme S, we know that Zariski locally we have $\{(E_i,e_i)/S_i\}_i$ defined by a Weierstrass equation (see Theorem 2.3.4). So we obtain $\{j_{(E_i,e_i)} \in \Gamma(S_i,\mathcal{O}_{S_i})\}_i$ and they lift to a global section $j_{(E,e)} \in \Gamma(S,\mathcal{O}_S)$ since by Proposition 2.5.1 they agree on the overlaps (regardless of the choice of Weierstrass equation). Hence we have canonically defined j-invariant, and we can therefore define a map

$$j\colon \mathcal{M}_{1,1} \to \mathbb{A}^1. \tag{2.30}$$

We want to apply Theorem 1.8.2 to assert that there exists a coarse moduli space $\pi\colon \mathcal{M}_{1,1} \to M_{1,1}$. We need the next proposition.

Proposition 2.5.2. $\mathcal{M}_{1,1}$ is an algebraically stack locally of finite presentation over $\mathbb{Z}$ with finite digonal.

Proof. Being locally of finite presentation is obvious. In fact by Proposition 2.3.5 we know that $U \to \mathcal{M}_{1,1}$ is smooth and surjective and by definition we only need to check that U is locally of finite presentation and this is trivial.

Next let us verify that the digonal of $\mathcal{M}_{1,1}$ is finite. Equivalently, if (E,e) and (E',e') are two elliptic curves over a scheme S, then the quasi-projective scheme $I = \underline{Isom}((E,e),(E',e'))$ is finite over S. The geometric fibers of this scheme are finite since an elliptic curve over an algebraically closed field has only finitely many automorphisms. By [Vak17], Theorem 29.6.2, it suffices to show that I is proper over S, which we do by verifying the valutative criterion for properness. To do so, assume that S is the spectrum of a discrete valuation ring with generic point $\nu \in S$ and closed point $s \in S$. Let

$$g_\nu\colon (E_\nu,e_\nu) \to (E'_\nu,e'_\nu)$$

be an isomorphism over the generic point. We must show that g_ν extends to an isomorphism over all of S. The fact that it extends, trivially follows from the "minimal model property" of elliptic curves, see [Sil09] Chapter 7.1. □

Corollary 2.5.3. The stack $\mathcal{M}_{1,1}$ is separated.

Proof. It is obvious since the diagonal is finite, hence proper. □

In particular, by Theorem 1.8.2, it follows that there exists the coarse moduli space for $\mathcal{M}_{1,1}$.
Since $\mathbb{A}^1$ is a scheme, the universal property of the coarse moduli space gives a unique factorization

$$\begin{tikzcd} \mathcal{M}_{1,1} \arrow[r, "\pi"] \arrow[rr, bend left, "j"] & M_{1,1} \arrow[r, "\bar{j}"] & \mathbb{A}^1. \end{tikzcd} \tag{2.31}$$

Theorem 2.5.4. The map $\bar{j}$ (2.31) is an isomorphism.

Proof. For this, note that the morphism $j\colon \mathcal{M}_{1,1} \to \mathbb{A}^1$ is proper. Again it suffices to verify the valutative criterion for properness. This amounts to showing that if V is a discrete valuation ring, and $(E,e)/K$ is an elliptic curve over the field of fractions K of V, such that the j-invariant $j_{(E,e)}$ is an element of V, then after possibly replacing V by a finite extension, there exists an elliptic curve over V whose generic fiber is (E,e). This follows from [Sil09], Chapter 7, Theorem 5.5.

It follows that the map $\bar{j}$ is also proper. Since it is quasi-finite it follows from [Vak17], Theorem 29.6.2, that the morphism $\bar{j}$ is also finite. Moreover, for every geometric point $t\colon Spec(k) \to \mathbb{A}^1$ the underlying set of $M_{1,1} \times_{\mathbb{A}^1} Spec(k)$ is a single point since elliptic curves over closed fields are classified by their j-invariant. This implies that $\bar{j}$ is birational (take t to be a geometric point lying over the generic point of $\mathbb{A}^1$). Since a birational finite morphism between integral schemes with the target normal is an isomorphism, we conclude that $\bar{j}$ is an isomorphism. □

Observation 2.5.5. Let k be a field. It is easy to prove with an analogous argument that the statement of the Theorem is also valid for $\mathcal{M}_{1,1,k}$. Instead, it is not possible to reuse the previous method to prove the theorem for general S. There is a proof of the previous Theorem in the case S is an affine noetherian scheme, see [FO10] Theorem 1.3. In the proof the authors bypass the request the scheme S being normal (that we need to conclude the proof) with another variant of the corollaries given by the Zariski Main Theorem.

However, in the rest of the section we use a different argument to generalize the Theorem in a wider context.

We want to consider the j-map obtained after the pull back along $S \to Spec(\mathbb{Z})$, i.e. $j\colon \mathcal{M}_{1,1,S} \to \mathbb{A}^1_S$. It would be a pleasant surprise if this was still the coarse moduli space for $\mathcal{M}_{1,1,S}$. This claim is actually correct, even if a priori this is not immediate, since the formation of coarse moduli spaces commutes (in general) only with flat base change, unless the stack is tame. But $\mathcal{M}_{1,1}$ is note tame, as we can see if take a geometric point $Spec(k) \to Spec(\mathbb{Z})$ with $char(k)$ equal to 2 or 3.

We want now to prove this result. First of all, observe that the j-invariant is equal to $j = c_4^3/\Delta$, hence the j-map can be extended to

$$j\colon \bar{\mathcal{M}}_{1,1} \to \mathbb{P}^1.$$

simply sending the nodal curves to ∞. With abuse of notation we named this map at the same manner of the map in 2.30. Notice that in this case too is possible to pull back the map along $S \to Spec(\mathbb{Z})$.

We will prove the following stronger result

Theorem 2.5.6. The morphism

$$j\colon \bar{\mathcal{M}}_{1,1,S} \to \mathbb{P}^1_S. \tag{2.32}$$

given by the j-invariant identifies $\mathbb{P}^1_S$ with the coarse moduli space of $\bar{\mathcal{M}}_{1,1,S}$.

The proof will be trivial at the end of this section after having proved Lemma 2.5.8. Before delving into this we present an immediate and fundamental Corollary.

Corollary 2.5.7. The morphism

$$j\colon \mathcal{M}_{1,1,S} \to \mathbb{A}^1_S. \tag{2.33}$$

given by the j-invariant identifies $\mathbb{A}^1_S$ with the coarse moduli space of $\mathcal{M}_{1,1,S}$.

Proof. Since $\mathbb{A}^1_S \to \mathbb{P}^1_S$ is flat and the square

$$\begin{tikzcd} \mathcal{M}_{1,1,S} \arrow[r, hook] \arrow[d, "j"] & \bar{\mathcal{M}}_{1,1,S} \arrow[d, "j"] \\ \mathbb{A}^1_S \arrow[r, hook] & \mathbb{P}^1_S \end{tikzcd}$$

is cartesian, the thesis trivially follow from Theorem 2.5.6 □

Theorem 2.5.6 is immediate by virtue of the following lemma.

Lemma 2.5.8. Let $\mathcal{X}$ be a Deligne-Mumford stack that is separated, flat and locally of finite type over $\mathbb{Z}$, and let

$$f\colon \mathcal{X} \to X$$

be its coarse moduli space map. If $f_{\mathbb{F}_p}\colon \mathcal{X}_{\mathbb{F}_p} \to X_{\mathbb{F}_p}$ is the coarse moduli space map of $\mathcal{X}_{\mathbb{F}_p}$ for every prime p, then $f_S\colon \mathcal{X}_S \to X_S$ is the coarse moduli space map of $\mathcal{X}_S$ for every scheme S.

Proof. The formation of the coarse moduli space $f\colon \mathcal{X} \to X$ commutes with flat base change in X and we may work fppf locally on X_S when checking that $f_S\colon \mathcal{X}_S \to X_S$ is the coarse moduli space of $\mathcal{X}_S$. In light of Theorem 1.8.4, we may therefore assume that $S = Spec(R)$ and that $X = Spec(A)$ and $\mathcal{X} = [Spec(B)/G]$ for some finite A-algebra B equipped with an action of a finite group G. In this situation, we have $A = B^G$, the coarse moduli space of $\mathcal{X}_S$ is $Spec((B \otimes_{\mathbb{Z}} R)^G)$. We want to prove that the map

$$j_R \colon B^G \otimes_{\mathbb{Z}} R \to (B \otimes_{\mathbb{Z}} R)^G$$

is an isomorphism, exploiting the fact that it is an isomorphism whenever $R = \mathbb{F}_p$ for any p (the reader can see the analogies with Corollary A.0.12 in the Appendix).

Since by hypothesis $\mathcal{X}$ is $\mathbb{Z}$-flat, it follows that B and B^G are torsion-free. Therefore, we have the inclusion $B^G \otimes_{\mathbb{Z}} R \to B \otimes_{\mathbb{Z}} R$, and hence also j_R, is injective for every $\mathbb{Z}$-module R. It remains to prove that j_R is also surjective.

By a passage to a filtered direct limit, we may assume that the $\mathbb{Z}$-module R is finitely generated. The case $R = \mathbb{Z}^n$ is obvious, so we may assume that $R = \mathbb{Z}/(n)$ for some $n \in \mathbb{N}_{\geq 1}$. We can decompose the problem through the short exact sequence

$$0 \longrightarrow \mathbb{Z}/(n_1) \longrightarrow \mathbb{Z}/(n) \longrightarrow \mathbb{Z}/(n_2) \longrightarrow 0.$$

Thanks to it, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B^G \otimes_{\mathbb{Z}} \mathbb{Z}/(n_1) \arrow[r] \arrow[d, hook, "j_{\mathbb{Z}/(n_1)}"] & B^G \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \arrow[r] \arrow[d, hook, "j_{\mathbb{Z}/(n)}"] & B^G \otimes_{\mathbb{Z}} \mathbb{Z}/(n_2) \arrow[r] \arrow[d, hook, "j_{\mathbb{Z}/(n_2)}"] & 0 \\
0 \arrow[r] & (B \otimes_{\mathbb{Z}} \mathbb{Z}/(n_1))^G \arrow[r] & (B \otimes_{\mathbb{Z}} \mathbb{Z}/(n))^G \arrow[r] & (B \otimes_{\mathbb{Z}} \mathbb{Z}/(n_2))^G. &
\end{tikzcd}$$

Using this, since we know the thesis in the case of $Z/(p)$ with p prime, we conclude by induction. □

Thanks to this last lemma, the proof of Theorem 2.5.6 becomes trivial, since $\mathcal{M}_{1,1}$ satisfies the hypothesis of the lemma (as we emphasized in the Observation 2.5.5).

CHAPTER 3

THE PICARD GROUP OF THE STACK $\mathcal{M}_{1,1,S}$

In this chapter we compute the Picard group of the moduli stack of elliptic curves. In order to state the main theorem of this thesis, we quickly re-explain the setting in which we work. However, we will adopt the same notation of Chapter 2 ([2.1]), so all the details can be found there.

As in the previous chapters, let $\mathcal{M}_{1,1}$ denote the moduli stack (over $\mathbb{Z}$) classifying elliptic curves, and for a scheme S let $\mathcal{M}_{1,1,S}$ denote the fiber product $S \times_{Spec(\mathbb{Z})} \mathcal{M}_{1,1}$.

Recall that on $\mathcal{M}_{1,1}$ there is the Hodge Bundle λ. For any morphism $t\colon T \to \mathcal{M}_{1,1}$ corresponding to an elliptic curve $f\colon E \to T$ the pullback $t^*\lambda$ is the line bundle $f_*\Omega^1_{E/T}$.

Note that if Λ is a ring, $t\colon Spec(\Lambda) \to \mathcal{M}_{1,1}$ is a morphism corresponding to an elliptic curve E/Λ, then after replacing Λ by a Zariski cover as in Theorem 2.3.4, the family E can be described by an equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

With these chosen coordinates a basis for $t^*\lambda$ is given by the *invariant differential*

$$\pi = \frac{dx}{2y + a_1x + a_3}.$$

Any two choices of coordinates differ by a transformation

$$\begin{aligned} x' &= u^2x + r, \\ y' &= u^3y + sx + t. \end{aligned} \tag{3.1}$$

where $u \in \Lambda^*$ and $r, s, t \in \Lambda$. One can easily compute that the invariant differential π' obtained from the coordinates (x', y') is equal to $u^{-1}\pi$. Instead the discriminant Δ' in the coordinates (x', y') is equal to $u^{12}\Delta$. In particular the element $\Delta\pi^{\otimes 12} \in t^*\lambda^{\otimes 12}$ is independent of the choice of coordinates, and therefore defines a trivialization of $\lambda^{\otimes 12}$ over $\mathcal{M}_{1,1}$. In particular it makes sense to consider the powers of $\lambda^{\otimes i}$ only modulo 12.

Let $p\colon \mathcal{M}_{1,1,S} \to \mathbb{A}^1_S$ the map defined by the j-invariant.

We will prove the following.

Theorem 3.0.1. Let S be a connected scheme. Then the map

$$\begin{aligned} \mathbb{Z}/12\mathbb{Z} \times \mathrm{Pic}(\mathbb{A}^1_S) &\longrightarrow \mathrm{Pic}(\mathcal{M}_{1,1,S}) \\ (i, \mathcal{L}) &\mapsto \lambda^{\otimes i} \otimes p^*\mathcal{L} \end{aligned} \tag{3.2}$$

is an isomorphism if either of the following hold:

(i) S is a $\mathbb{Z}[1/2]$-scheme.

(ii) S is reduced.

Remark 3.0.2. As we observe in 3.6.3, the theorem fails for nonreduced schemes in characteristic 2.

Remark 3.0.3. If the base scheme S is not connected, the Picard group is simply $\prod_{i\in I} Pic(\mathcal{M}_{1,1,S_i})$, where the S_i's are the connected components of S.

3.1. When 6 is invertible on S

Though the case when 6 is invertible follows from the more technical work in subsequent sections, we include here a proof in the case of a $\mathbb{Z}[1/6]$-scheme since it is much easier than the more general cases.

Let $\mathrm{Proj}(\mathbb{Z}[x,y,z][1/6]/(y^2z = x^3 + xz^2)) \otimes_{Spec(\mathbb{Z})} S \to S$ be the elliptic curve with automorphism group μ_4 ($\Delta = -64$, $j = 1728$). Let $\tilde{s}_4 \colon S \to \mathcal{M}_{1,1,S}$ be the section corresponding through the Yoneda Lemma to that elliptic curve. This section defines closed immersion $s_4 \colon B\mu_{4,S} \hookrightarrow \mathcal{M}_{1,1,S}$.

Similarly, let $\mathrm{Proj}(\mathbb{Z}[x,y,z][1/6]/(y^2z + yz^2 = x^3)) \otimes_{Spec(\mathbb{Z})} S \to S$ be the elliptic curve with automorphism group μ_6 ($\Delta = -27$, $j = 0$) and $\tilde{s}_6 \colon S \to \mathcal{M}_{1,1,S}$ be the corresponding section. Likewise, it defines closed immersion $s_6 \colon B\mu_{6,S} \hookrightarrow \mathcal{M}_{1,1,S}$. For any line bundle $\mathcal{L}$ on $\mathcal{M}_{1,1,S}$ the pullback $s_4^*\mathcal{L}$ (respectively $s_6^*\mathcal{L}$) corresponds to a line bundle M_4 (respectively M_6) on S with action of the group μ_4 (respectively μ_6). We thus get maps

$$\rho_4 \colon \mu_4 \to \underline{Aut}(M_4) \simeq \mathbb{G}_m, \quad \rho_6 \colon \mu_6 \to \underline{Aut}(M_6) \simeq \mathbb{G}_m, \tag{3.3}$$

defining characters $\chi_4 \in \mathbb{Z}/4\mathbb{Z}$ and $\chi_6 \in \mathbb{Z}/6\mathbb{Z}$. Observe that here we are using the hypothesis that S is connected to obtain the isomorphism $\underline{Aut}(M_4) \simeq \underline{Aut}(M_4) \simeq \mathbb{G}_m$.

Lemma 3.1.1. The pair $(\chi_4\,, \chi_6)$ lies in $\mathbb{Z}/12\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

Proof. The construction of the pair $(\chi_4\,, \chi_6)$ commutes with arbitrary base change on S so it suffices to consider the case when S is the spectrum of an algebraically closed field $S = Spec(k)$. We have to show that $\rho_4|_{\mu_2} = \rho_6|_{\mu_2}$.

Write $k[[t]]$ for the completion of the local ring of $\mathbb{A}^1$ at $j = 1728$ and let $k[[z]]$ be the completion of the local ring of $\mathcal{M}_{1,1,S}$ at the point corresponding to the curve $y^2 = x^3 + x$. To see how the map $k[[t]] \to k[[z]]$ works, we can for example take the Legendre family $Spec(k[\lambda][1/(\lambda)(\lambda-1)])$ which provides an étale cover of $\mathcal{M}_{1,1,k}$. Therefore, taking the localization of this family at the right point, we actually obtain $k[[z]]$. The map is

$$\begin{aligned} k[t] &\to k[\lambda][1/(\lambda)(\lambda-1)] \\ t &\to 2^8 \frac{(t^2-t+1)^3}{t^2(t-1)^2} \end{aligned}$$

and this map is $6:1$ except that in $j = 1728$ and $j = 0$ in which it is respectively $3:1$ and $2:1$. These details about the Legendre family are standard and so they are omitted. For a reference see [Sil09], Chapter 3.1 in the section about the Legendre form. Putting all together we deduce that, after a suitable translation, the map between local rings $k[[t]] \to k[[z]]$ sends t to z^2.

Now we want to understand how the group μ_4 acts on $k[[z]]$. The claim is that $\zeta_4 \star z = \zeta_4^2 z = -z$. For proving this, we use a different presentation.

We have already seen that when S is a $\mathbb{Z}[1/6]$-scheme, every elliptic curve is locally given by a Weierstrass equation of the form $y^2 = x^3 + Ax + B$, hence $\mathcal{M}_{1,1}$ can be presented as the stack theoretic quotient $[(Spec(k[x,y][1/\Delta])/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts via $u(a,b) = (u^4a, u^6b)$. We are interested in the point $\bar{w} := (A,B) = (1,0)$. We want to find a slice for $\bar{w}$, that is a couple $((W,w),\varphi)$ where (W,w) is a pointed affine scheme such that the automorphism group $\Gamma_{\bar{w}}$ (of $\bar{w}$ in $\mathcal{M}_{1,1,k}$) acts on W fixing w and where φ is an étale map $\varphi\colon [W/\Gamma_{\bar{w}}] \to \mathcal{M}_{1,1,k} = [(Spec(k[x,y][1/\Delta])/\mathbb{G}_m]$ such that $\varphi(w) = \bar{w}$. In this way we can compute the action of $\Gamma_{\bar{w}}$ on W.

We can consider the family $y^2 = x^3 + x + z$ (so $A = 1$ and $B = z$ left free to vary). In other words, we have the morphism $[A^1/\mathbb{G}_m] \to [(Spec(k[x,y][1/\Delta])/\mathbb{G}_m]$, and it is indeed a slice. Note that the morphism is well defined: a computation, see [Sil09] Remark 4.1.3 for the details, shows that $\Delta = -16(4A^3 + 27B^2)$ and therefore $A = 1 \implies \Delta \neq 0$.

The restriction of the action of $\mathbb{G}_m$ to this family is simply $u \star z = u^6 z$. In particular the action of ζ_4 is $\zeta_4 \star z = \zeta_4^2 z = -z$, and so localizing at z we have proved the claim.

Furthermore, we can write $\mathcal{L}|_{k[[z]]} = k[[z]] \cdot e$ for some basis e, such that ρ_4 acts by $\zeta_4 \cdot e = \zeta_4^{\chi_4} e$. From this we see that $\rho_4|_{\mu_2}$ is equal to the character defined by the action of μ_2 on the fiber of $\mathcal{L}$ at the generic point of $\mathcal{M}_{1,1,S}$. Similarly, $\rho_6|_{\mu_2}$ is equal to the action on the generic fiber.

□

We therefore obtain a map

$$\begin{aligned} Pic(\mathcal{M}_{1,1,S}) &\to \mathbb{Z}/12\mathbb{Z} \\ \mathcal{L} &\mapsto (\chi_4\,,\,\chi_6), \end{aligned} \tag{3.4}$$

and it follows from the construction that this map is a homomorphism. Let K denote the kernel.

We prove now a technical lemma that we will need later.

Lemma 3.1.2. Let $\mathcal{X}$ be a tame Deligne-Mumford stack with coarse moduli space $\pi\colon \mathcal{X} \to X$. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$ such that for every geometric point $\tilde{x} \to \mathcal{X}$ the action of the stabilizer group $G_{\tilde{x}}$ on $\mathcal{L}_{\tilde{x}}$ is trivial. Then $\pi_*\mathcal{L}$ is an invertible sheaf on X and $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is an isomorphism.

Proof. It suffices to prove the lemma after passing to the strict henselization of X at the geometric point $\tilde{x}$. Let $A = \mathcal{O}_{X,\tilde{x}}$ and $B = \mathcal{O}_{\mathcal{X},\tilde{x}}$. Then, as explained in Theorem 1.8.4, if Γ denotes the stabilizer group of $\tilde{x}$ then there is a natural action of Γ on B such that $\mathcal{X} = [Spec(B)/\Gamma]$. Let M be the free B-module with Γ-action of rank 1 defining $\mathcal{L}$. M is free because B is a local ring. By hypothesis, since $\mathcal{X}$ is tame, the group Γ is linearly reductive. In particular, it follows that the representation category of Γ is semisimple. Moreover, by our assumptions, Γ acts in the trivial way on $M \otimes k(\tilde{x})$, and therefore it is generated by an invariant element and choosing a lifting to an invariant element of M we see that we can write $M = B \cdot e$ where Γ acts trivially on e. Then, by definition of pushforward of a sheaf over a stack into the coarse moduli space, $\pi_*\mathcal{L}$ is just $A \cdot e$ and the lemma is immediate. □

Corollary 3.1.3. The homomorphism $\pi^*\colon Pic(\mathbb{A}^1_S) \to K$ is an isomorphism.

Proof. First of all, observe that if $\mathcal{W} \in Pic(\mathbb{A}^1_S)$ then $\pi^*(\mathcal{W}) \in K$. This is obvious because by definition the pullback of a sheaf on $\mathbb{A}^1_S$ to $\mathbf{B}\mu_i$ is a sheaf on S which is invariant if pulled back through pr_2 or ρ (we mean the action of G on U) into $\mu_i \times S$.

We show that if $\mathcal{L}$ is a line bundle with $(\chi_4\,,\,\chi_6) = (0,\,0)$, i.e. $\mathcal{L} \in K$, then $\pi_*\mathcal{L}$ is an invertible sheaf on $\mathbb{A}^1_S$ and $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. Observe that if S is a $\mathbb{Z}[1/6]$-scheme, then the automorphism groups of the geometric points are μ_6, μ_4 or μ_2, and this groups are linearly reductive since they are actually tame groups; so for Theorem A.0.11 we know that $\mathcal{M}_{1,1,S}$ is a tame stack. By Lemma 3.1.2 it suffice to show that for any geometric point $\tilde{x} \to \mathcal{M}_{1,1,S}$ the action of the stabilizer group $\tilde{x}$ on $\mathcal{L}(\tilde{x})$ is trivial. For this we may assume that S is the spectrum of an algebraically closed field. By our assumptions the actions ρ_4 and ρ_6 are trivial. By the argument used in the proof of 3.1.1 this implies that the action of the generic stabilizer is also trivial, and so we have the thesis. □

What we need now to conclude is to prove the following lemma.

Lemma 3.1.4. The image of λ (The Hodge Bundle) in $\mathbb{Z}/12\mathbb{Z}$ is a generator. In particular 3.4 is surjective.

Proof. It suffices to consider the case when S is the spectrum of a field in which case the above shows that $Pic(\mathcal{M}_{1,1,S})$ injects into $\mathbb{Z}/12\mathbb{Z}$. We can in fact compute directly the image of λ in $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

The image in $\mathbb{Z}/4\mathbb{Z}$ corresponds to the representation of μ_4 given by the action on the invariant differential $\frac{dx}{2y}$ of the curve $y^2 = x^3 + x$. An element $\zeta_4 \in \mu_4$ acts by $(x, y) \mapsto (\zeta_4^2 x, \zeta_4 y)$ and therefore the action on $\frac{dx}{2y}$ is equal to multiplication by ζ_4. Therefore the image of λ in $\mathbb{Z}/4\mathbb{Z}$ is equal to 1.

Similarly, the image of λ in $\mathbb{Z}/6\mathbb{Z}$ corresponds to the character given by the invariant differential $\frac{dx}{2y+1}$ of the curve $y^2 + y = x^3$. Write $\mu_6 = \mu_2 \times \mu_3$. Then $(\zeta_2, 1)$ acts by $(x, y) \mapsto (x, -y-1)$ and $(1, \zeta_3)$ acts by $(x, y) \mapsto (\zeta_3 x, y)$. Therefore $(\zeta_2, 1)$ acts on the invariant differential by multiplication by -1 and $(1, \zeta_3)$ by multiplication by ζ_3. It follows that λ maps to $1 \in \mathbb{Z}/6\mathbb{Z}$ which implies that λ is a generator in $\mathbb{Z}/12\mathbb{Z}$.

□

Corollary 3.1.5. The map $\lambda \times \pi^*\colon (\mathbb{Z}/12\mathbb{Z}) \times Pic(\mathcal{A}^1_S) \to Pic(\mathcal{M}_{1,1,S})$ is an isomorphism if S is a $\mathbb{Z}[\frac{1}{6}]$-scheme.

3.2. The case of a normal affine scheme S

In all this section we only consider the base scheme S to be affine and normal. So, Write $S = Spec(\Lambda)$ with Λ a normal affine ring. For clarity of exposition we repeat the notation fixed in the Theorem 2.3.5. Let U be the scheme

$$U := Spec(\Lambda[a_1, a_2, a_3, a_4, a_6, \frac{1}{\Delta}]),$$

Let G denote the group scheme with underlying scheme $Spec(\Lambda[u^{\pm}, r, s, t])$ with group law defined by

$$(u', r', s', t') \cdot (u, r, s, t) = (uu', u^2r' + r, us' + s, u^3t' + u^2r's + t).$$

Then $\mathcal{M}_{1,1,S}$ is isomorphic to the stack theoretic quotient $[U/G]$ (see Theorem 2.3.5).

Lemma 3.2.1 (Homotopic invariance for Pic)**.** Let $S = Spec(\Lambda)$ be a normal noetherian affine ring. Then

$$Pic(S) \cong Pic(\mathbb{A}^n_S).$$

Proof. It suffices to prove the case with $n = 1$ and after the thesis is immediate by induction.

Consider the diagram

$$\begin{array}{ccc} Pic(S) & \longrightarrow & Pic(\mathbb{A}^1_S) \\ \downarrow & & \downarrow \\ Cl(S) & \longrightarrow & Cl(\mathbb{A}^1_S), \end{array}$$

where Cl represents the class group, i.e. the group Div modulo linear equivalence. First of all, observe that the map between class groups is an isomorphism. This follows from the fact there are two types of height one prime ideals in $\mathbb{A}^1_S$: (x) and the height one prime ideal $\mathfrak{p}$ of $Spec(\Lambda)$. But (x) is principal, so equal to zero in $Cl(\mathbb{A}^1_S)$. From the normality we also know that the vertical arrows are injective, and so it follows that the map between Picard groups is injective too.

Now we want to say that it is surjective. Notice that it is well defined the map $CaDiv(S) \to CaDiv(\mathbb{A}^1_S)$ because the map is faithfully flat (see [Gro67] Proposition 21.4.5). Moreover, we can apply [Gro67] Corollary 21.4.11 at page 360 for concluding this map is surjective. □

We need now a technical lemma. We present it without a proof to avoid interrupting the flow of the discussion, and we prove it without the details in the appendix.

Theorem 3.2.2. Let Λ be a normal domain, then it is the filtered colimit of a system consisting of finite type and normal subrings.

Proof. For a proof, see the Appendix B. □

Lemma 3.2.3. Let $\Delta \in \mathbb{Z}[t_1, \dots, t_n]$ be a polynomial satisfying

(i) The greatest common divisor of the coefficients of its non constant monomials is 1.

(ii) For any field k the image of Δ in $k[t_1, \dots, t_n]$ is irreducible.

Then for any normal ring Λ, the pullback homomorphism

$$Pic(\Lambda) \to Pic(\Lambda[t_1, \dots, t_n, \frac{1}{\Delta}]) \tag{3.5}$$

is an isomorphism.

Proof. First of all, by 3.2.2 we can approximate Λ with normal noetherian domains. For filtered colimits

$$Spec(\Lambda) = \varprojlim Spec(\Lambda_i)$$

we have that

$$Pic(Spec(\Lambda)) = Pic(Spec(\varinjlim \Lambda_i)) = Pic(\varprojlim Spec(\Lambda_i)) = \varinjlim Pic(Spec(\Lambda_i)).$$

The analogous chain of equalities holds true for $\Lambda[t_1, \ldots, t_n, \frac{1}{\Delta}]$ and so it is sufficient to deal with the spectra of normal noetherian domains.

Hence suppose that Λ is noetherian. The assumption (i) implies that the map $Spec(\mathbb{Z}[t_1, \ldots, t_n][\frac{1}{\Delta}]) \to Spec(\mathbb{Z})$ is surjective (for example because the ideal (p) continues to be prime in $\mathbb{Z}[t_1, \ldots, t_n][\frac{1}{\Delta}]$ since $p \nmid \Delta$). In particular this map is faithfully flat. It follows that the map

$$Spec(\Lambda[t_1, \ldots, t_n][\frac{1}{\Delta}]) \to Spec(\Lambda)$$

is also faithfully flat.

By the preceding observation the divisor $V(\Delta) \subset Spec(\Lambda[t_1, \ldots, t_n])$ is irreducible. In fact if it were factored as the product of two non-invertible polynomial than it would not be irreducible in $\bar{k}[t_1, \ldots, t_n]$ with $\Lambda \subset \bar{k}$, which contradicts (ii).

It follows that there is an exact sequence of Weil divisor class group

$$\mathbb{Z}[V(\Lambda)] \to Cl(\Lambda[t_1, \ldots, t_n]) \to Cl(\Lambda[t_1, \ldots, t_n][\frac{1}{\Delta}]) \to 0, \tag{3.6}$$

where the first arrow is the zero map. We conclude that

$$Cl(\Lambda) \cong Cl(\Lambda[t_1, \ldots, t_n]) \cong Cl(\Lambda[t_1, \ldots, t_n][\frac{1}{\Delta}]). \tag{3.7}$$

The normality of Λ implies that the natural maps from the Picard groups to the Weil divisor class groups are injective. Thus it suffices to show that if $D \in Cl(\Lambda)$ is a Weil divisor whose image in $Cl(\Lambda[t_1, \ldots, t_n][\frac{1}{\Delta}])$ is in the image of $Pic(\Lambda[t_1, \ldots, t_n][\frac{1}{\Delta}])$, then D is obtained from a line bundle on $Spec(\Lambda)$. This follows from the observation that $\Lambda \to \Lambda[t_1, \ldots, t_n][\frac{1}{\Delta}]$ is faithfully flat, from the isomorphism between $Pic(\Lambda)$ and $Pic(\Lambda[t_1, \ldots, t_n])$ and from Theorem 6 - chapter 4 in [isc]. □

We come back now to our main setting.

Proposition 3.2.4. The pullback map

$$Pic(S) \to Pic(U) \tag{3.8}$$

is an isomorphism.

Proof. We apply the previous lemma to $\Delta \in \mathbb{Z}[a_1, \ldots, a_6]$. Then (i) and (ii) are immediate. □

The isomorphism $\mathcal{M}_{1,1,S} \cong [U/G]$ defines a morphism $\sigma\colon \mathcal{M}_{1,1,S} \to \mathbf{B}G$, in the obvious way.

For a character $\chi\colon G \to \mathbb{G}_m$ defining a line bundle on $\mathbf{B}G$ (according to Theorem 1.10.1), let L_χ be the line bundle on $\mathcal{M}_{1,1,S}$ obtained by pulling back along σ.

Lemma 3.2.5. Let $\mathcal{L}$ be a line bundle on $\mathcal{M}_{1,1,S}$ such that the pullback L of $\mathcal{L}$ to U is trivial. Then $\mathcal{L} \cong L_\chi$ for some character $\chi\colon G \to \mathbb{G}_m$.

Proof. Fix a basis $e \in L$.

Let $\mathcal{F}$ be the sheaf on the category of affine S-schemes (with the étale topology) which to any morphism of affine schemes $S' \to S$ associates $\Gamma(U_{S'}, \mathcal{O}^*_{U_{S'}})$. There is an inclusion of sheaves $\mathbb{G}_m \subset \mathcal{F}$ given by the inclusions $\Gamma(S', \mathcal{O}^*_{S'}) \subset \Gamma(U_{S'}, \mathcal{O}^*_{U_{S'}})$. For any $S' \to S$ and $g \in G(S')$, we get an element $u_g \in \mathcal{F}(S')$ defined by the condition that $g(e) = u_g \cdot e \in L$. This defines a map of sheaves (not necessarily a homomorphism)

$$f\colon G \to \mathcal{F}. \tag{3.9}$$

Suppose we have a map $\tau\colon U \to \mathbf{B}G$ and a quasi-coherent sheaf $\mathcal{G}$ on $\mathbf{B}G$. The map τ factors through $\vartheta\colon U \to S$. The pullback of $\mathcal{G}$ via $S \to \mathbf{B}G$ is a G-equivariant quasi-coherent sheaf V on S. In other words a sheaf V together with a homomorphism $G \to \underline{Aut}_\Lambda(V) \cong \mathbb{G}_m$. Then the pullback $\tau^*\mathcal{G}$ is exactly the sheaf ϑ^*V together with the data of $G \to \mathbb{G}_m$.

Saying that $\mathcal{L} \cong L_\chi$ is equivalent to say that L is a sheaf on U together with an additional representation $G \to \underline{Aut}_\Lambda(V) \cong \mathbb{G}_m$.Hence to prove the lemma it suffices to show that f has image contained in $\mathbb{G}_m \subset \mathcal{F}$. Before starting the proof of the claim, note that it is clear that if this holds than the map $G \to \mathbb{G}_m$ is a homomorphism. In fact, consider the diagram:

$$\begin{tikzcd} G \times G \cong Spec(\Lambda[u_1^{\pm}, r_1, s_1, t_1]) \times Spec(\Lambda[u_2^{\pm}, r_2, s_2, t_2]) \arrow[r, "f\times f"] \arrow[d, "m"] & \mathbb{G}_m \times \mathbb{G}_m \arrow[d, "\hat{m}"] \\ Spec(\Lambda[u^{\pm}, r, s, t]) \arrow[r, "f"] & \mathbb{G}_m. \end{tikzcd} \tag{3.10}$$

Since G is affine the map f is determined by a section $\gamma \in \mathbb{G}_m(G) = \{c \cdot u^m | c \in \Lambda^*\}$, and it is easy to observe that the diagram commute.

Now we prove the claim. Since G is an affine scheme the map f is determined by a section $u_0 \in \mathcal{F}(G)$. Since G is normal and connected, this section $u_0 \in \Gamma(U_G, \mathcal{O}^*_{U_G})$ can be written uniquely as $\beta\Delta^m$, where $\beta \in \Gamma(G, \mathcal{O}^*_G)$ and $m \in \mathbb{Z}$. We need to show that $m = 0$. Let consider the following diagram:

$$\begin{tikzcd} U \arrow[r, "\tilde{e}"] \arrow[d] & U_G \arrow[r] \arrow[d] & U \arrow[d] \\ S \arrow[r, "e"] & G \arrow[r] & S \end{tikzcd}$$

Note that the image of u_0 under the map $\mathcal{F}(G) \to \mathcal{F}(S)$ defined by the identity section $e\colon S \to G$ is equal to 1. It follows that $\tilde{e}^\sharp(\beta \cdot \Delta^m) = e^\sharp(\beta) \cdot \Delta^m$ is equal to 1 in $\Gamma(U, \mathcal{O}^*_U)$ which implies that $m = 0$. ☐

Observe that thanks to the considerations about the diagram 3.10 in the last proof, we have also proven the following crucial lemma.

Lemma 3.2.6. Any homomorphism $G \to \mathbb{G}_m$ factors through the projection

$$\begin{aligned} \chi_0\colon G &\to \mathbb{G}_m \\ (u, r, s, t) &\mapsto u. \end{aligned} \tag{3.11}$$

We also need the following lemma.

Lemma 3.2.7. Let $n, m \in \mathbb{Z}$ be integers. Then $L_{\chi_0^n} \cong L_{\chi_0^m}$ if and only if $n \equiv m \ (mod\, 12)$.

Proof. Observe that $L_{\chi_0^n} \otimes L^{-1}_{\chi_0^m} \cong L_{\chi_0^{n-m}}$, so it suffices to show that $L_{\chi_0^n} \cong \mathcal{O}_{\mathcal{M}_{1,1,S}}$ if and only if $12|n$.

Choose a basis $e \in L_{\chi_0^n}(U)$ such that G acts on e through χ_0^n. Then an isomorphism $L_{\chi_0^n} \cong \mathcal{O}_{\mathcal{M}_{1,1,S}}$ is given by a function $\beta \in \Gamma(U, \mathcal{O}^*_U)$ such that the action of G on $\beta^{-1} \cdot e$ is trivial. Equivalently, we want a global section $\beta \in \Gamma(U, \mathcal{O}^*_U)$ such that G acts on β through χ_0^n. Since, as we have already seen,

$$\Gamma(U, \mathcal{O}^*_U) \cong \Lambda^* \cdot \Delta^{\mathbb{Z}}$$

and G acts on Δ through χ_0^{12}, such a unit exists if and only if $12|n$. ☐

Now we want to put all these results together.

Consider the map $Pic(\mathcal{M}_{1,1,S}) \to Pic(U)$. This map is surjective (indeed, given a line bundle on U, to obtain a line bundle on $\mathcal{M}_{1,1,S}$ we only need the further information of the action of G, and we can simply impose this action being trivial).

By 3.2.4, if $\mathcal{L}$ is a line bundle on $\mathcal{M}_{1,1,S}$ then the pullback of $\mathcal{L}$ to U is isomorphic to the pullback of a line bundle M on $\mathbb{A}^1_S$. It follows that any line bundle on $\mathcal{M}_{1,1,S}$ is isomorphic to $M \otimes L_\chi$ for some character $\chi\colon G \to \mathbb{G}_m$. More such a line bundle $M \otimes L_\chi$ is trivial if and only if M is trivial and L_χ is trivial.

The line bundle λ is trivialized over U by the invariant differential π and the action of $(u, r, s, t) \in G$ on π is through the character $G \to \mathbb{G}_m$ sending (u, r, s, t) to u^{-1}. So we obtain that the map 3.2 is an isomorphism.

3.3. Without the assumption of S being normal

If $\mathcal{L}$ is a line bundle on $\mathcal{M}_{1,1,S}$, there is a unique function $s \mapsto l(s) \in \mathbb{Z}/12\mathbb{Z}$ which associates to a point $s\colon Spec(k(s)) \to S$ the unique power $l(s)$ of λ such that $\mathcal{L}_s \otimes \lambda^{-l(s)}$ on $\mathcal{M}_{1,1,k(s)}$ descends to $\mathbb{A}^1_{k(s)}$. Here we are using that we know the thesis for $k(s)$ since we have proven the normal case in the previous section.

Lemma 3.3.1. The function $s \mapsto l(s)$ is a locally constant function on S. Here locally means simply with respect to the topology of S.

Proof. The assertion is local on S so we may assume that S is affine. Every morphism $Spec(k) \to Spec(A)$ factors through $Spec(A^{red})$ (the reduced structure). Thus, we can assume the scheme to be reduced (the underlying topology is the same). Furthermore, the assertion can be verified on each irreducible component since any two irreducible closed sets on a connected component have non trivial intersection. So we may assume that S is integral. Finally, if $\tilde{S} \to S$ is the normalization then it suffices to verify the assertion for $\tilde{S}$. In this case the result follows from section 3.2. □

In particular if S is connected we obtain a homomorphism

$$\Psi\colon Pic(\mathcal{M}_{1,1,S}) \to \mathbb{Z}/12\mathbb{Z} \tag{3.12}$$

sending λ to 1.

We say that a scheme S has the ♣ property if it satisfies the following condition:

For each line bundle $\mathcal{L}$ on $\mathcal{M}_{1,1,S}$ such that for every field valued point $s\colon Spec(k(s)) \to S$ the sheaf $\mathcal{L}|_{\mathcal{M}_{1,1,k(s)}}$ is pulled-back from a sheaf $L|_{\mathbb{A}^1_{k(s)}}$ on $\mathbb{A}^1_{k(s)}$, then there exists a unique line bundle L, up to isomorphism, on $\mathbb{A}^1_S$ such that $\pi^*L \cong \mathcal{L}$.

In other words, a scheme S has the ♣ property if and only if the kernel of 3.12 is isomorphic to $Pic(\mathbb{A}^1_S)$.

Theorem 3.2 says exactly that reduced connected schemes and connected $\mathbb{Z}[1/2]$-schemes have the ♣ property. Before proceeding to prove Theorem 3.2 we need to simplify the problem.

Lemma 3.3.2. Let S be a scheme. Suppose that every $S_i = Spec(R_i) \subset S$ (open affine subscheme of S) has the property ♣. Then S has the property ♣.

Proof. Let $\mathcal{L}$ be a line bundle on $\mathcal{M}_{1,1,S}$ such that for every field valued point $s\colon Spec(k(s)) \to S$ the sheaf $\mathcal{L}|_{\mathcal{M}_{1,1,k(s)}}$ is pulled-back from a sheaf $L|_{\mathbb{A}^1_{k(s)}}$ on $\mathbb{A}^1_{k(s)}$. We want to construct a line bundle L on $\mathbb{A}^1_S$ whose pull back to $\mathcal{M}_{1,1,S}$ is $\mathcal{L}$.

Let $\{S_i\}_i$ be an affine cover of S. If $s\colon Spec(k(s)) \to S$ is a point which factors through S_i we have the following situation

$$\begin{array}{ccccc}
\mathcal{M}_{1,1,k(s)} & \xrightarrow{\pi_s} & \mathbb{A}^1_{k(s)} & \longrightarrow & Spec(k(s)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_{1,1,S_i} & \xrightarrow{\pi_i} & \mathbb{A}^1_{S_i} & \longrightarrow & S_i \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_{1,1,S} & \xrightarrow{\pi} & \mathbb{A}^1_S & \longrightarrow & S.
\end{array}$$

Since $\{S_i\}_i$ is an open cover for S, thus $\{\mathbb{A}^1_{S_i}\}$ is an open cover for $\mathbb{A}^1_S$. Since the S_i's are affine, by hypothesis they have the ♣ property. Hence there exists a unique line bundle L_i on $\mathbb{A}^1_{S_i}$ such that $\pi_i^*(L_i) = \mathcal{L}_i := \mathcal{L}|_{\mathcal{M}_{1,1,S_i}}$. Moreover, notice that there is a natural isomorphism $L_i \cong \pi_{i*}(\mathcal{L}_i) = \pi_{i*}\pi_i^*(L_i)$.

Suppose that such an L exists. Then it is straightforward unique since $L|_{\mathbb{A}^1_{S_i}} \cong L_i$ by the uniqueness of L_i.

For the existence of such an L, we claim that the L_i's glue together. It is clear that if this happens, then $\pi^*L \cong \mathcal{L}$ since this isomorphism is a local property and it can be verified on each S_i.

Observe that since $\mathcal{L}$ is a line bundle, there is a natural isomorphism

$$\Psi_{ij}\colon \mathcal{L}_i|_{\mathcal{M}_{1,1,S_i\cap S_j}} \cong \mathcal{L}_j|_{\mathcal{M}_{1,1,S_i\cap S_j}}.$$

Hence, exploiting the fact that $L_i \cong \pi_{i*}(\mathcal{L}_i)$ for all i, we can push forward the isomorphism Ψ_{ij} via π to obtain an isomorphism

$$\pi_{i*}\Psi_{ij} = \Phi_{ij}\colon L_i|_{\mathbb{A}^1_{S_i\cap S_j}} \xrightarrow{\cong} L_j|_{\mathbb{A}^1_{S_i\cap S_j}}.$$

Furthermore, it is clear that it satisfies the cocycle condition since for Ψ_{ij} the condition holds. Hence the L_i's glue together and this guarantees the formation of the line bundle L over $\mathbb{A}^1_S$.

□

Observation 3.3.3. It follows from the previous lemma that if we want to prove the Theorem 3.0.1 in the case of S connected and reduced, we can suppose S connected, reduced and affine too. In the same way, when S is a connected $\mathbb{Z}[1/2]$-scheme we can suppose to be affine too. Moreover, we can suppose $S = Spec(R)$ to be noetherian. Actually, once the thesis is proven for noetherian schemes, we obtain the general result in the following way:

we can write $Spec(R) = \varprojlim Spec(R_\alpha) = \varprojlim S_\alpha$ with R_α noetherian. We obtain the diagram

$$\begin{tikzcd}
\mathcal{M}_{1,1,k(s)} \arrow[r, "\pi_s"] \arrow[d, "g"] & \mathbb{A}^1_{k(s)} \arrow[r] \arrow[d] & Spec(k(s)) \arrow[d] \\
\mathcal{M}_{1,1,S} \arrow[r, "\pi"] \arrow[d, "f_\alpha"] & \mathbb{A}^1_S \arrow[r] \arrow[d, "h_\alpha"] & S \arrow[d] \\
\mathcal{M}_{1,1,S_\alpha} \arrow[r, "\pi_\alpha"] & \mathbb{A}^1_{S_\alpha} \arrow[r] & S_\alpha.
\end{tikzcd}$$

All the squares are cartesian. Given a line bundle $\mathcal{L}$ on $\mathcal{M}_{1,1,S}$, there exists a β and a $\mathcal{O}_{\mathcal{M}_{1,1,S_\beta}}$-sheaf $\mathcal{L}_\beta$, such that $\mathcal{L} \cong f_\beta^*\mathcal{L}_\beta$. Moreover, $\mathcal{L} \cong \varinjlim \mathcal{L}_\alpha$ where $\mathcal{L}_\alpha = f^*_{\alpha,\beta}\mathcal{L}_\beta$ for all $\alpha \geq \beta$. For these results see [Gro66], Paragraph 8.2.

In particular, if we know that S_β has the ♣ property, then there exists L_β line bundle on $\mathbb{A}^1_{S_\beta}$ such that $\pi_\beta^* L_\beta = \mathcal{L}_\beta$ and therefore $h_\alpha^* L_\beta$ is the line bundle from which $\mathcal{L}$ descents.

3.4. The case when S is reduced

In this section we always assume that S is a reduced connected scheme.

In order to prove the Theorem 3.0.1, it remains to show that the Kernel of the map 3.12 is equal to $Pic(\mathbb{A}^1_S)$ (following the notation of the previous section it means that reduced connected schemes have the ♣ property). By Lemma 3.3.2 and Observation 3.3.3, it suffices to prove the thesis for an affine connected noetherian reduced scheme $S = Spec(R)$.

To complete the proof of 3.2 in the case when S is reduced, we make some general observations about the relationship between line bundles on a stack and line bundles on the coarse moduli space.

Let S be a noetherian scheme and $\mathcal{X} \to S$ be a Deligne-Mumford stack over S. Let $\pi\colon \mathcal{X} \to X$ be the coarse moduli space, and assume that the formation of the coarse space X commutes with arbitrary base change on S and that X is reduced (we just saw that this holds for $\mathcal{M}_{1,1,S}$ over a reduced scheme). For a field valued point $x\colon Spec(k) \to S$ let $\pi_x\colon \mathcal{X}_x \to X_x$ denote the base change $\mathcal{X} \times_S x \to X \times_S x$.

Proposition 3.4.1. Let L be a line bundle on $\mathcal{X}$ such that for every field valued point $x\colon Spec(k) \to S$ the sheaf $\pi_{x*}(L|_{\mathcal{X}_x})$ is locally free of rank 1 and $(\pi_x)^*(\pi_x)_*(L|_{\mathcal{X}_x}) \to L|_{\mathcal{X}_x}$ is an isomorphism. If $\mathcal{X} \to X$ is flat, then the sheaf π_*L is locally free of rank 1 on X and $\pi^*\pi_*L \to L$ is an isomorphism.

Proof. One immediately reduces to the case when $X = Spec(R)$, $Y = Spec(B)$ is a finite flat R scheme, Γ is a finite group acting on Y over X such that $\mathcal{X} = [Y/\Gamma]$ (indeed étale locally on the coarse space every Deligne-Mumford stack can be presented in this way (Proposition 1.8.4). Let M denote the B-module corresponding to L, so that M comes equipped with an action of Γ over the action on B. We can even assume that R is a local ring and that M is a free R-module (forgetting the B-module structure). We are then trying to compute the kernel of the map

$$\begin{aligned} M &\to \prod_{\iota\in\Gamma} M, \\ m &\mapsto (\dots, \iota(m) - m, \dots)_{\iota\in\Gamma}. \end{aligned} \tag{3.13}$$

We can also assume that $S = Spec(\Lambda)$ is affine.

Lemma 3.4.2. Let R be a reduced local Λ-algebra and let $A \in M_{n\times m}(R)$ be a matrix (which we view as a map $R^n \to R^m$) with the property that for every $x \in Spec(\Lambda)$ the matrix $A(x) \in M_{n\times m}(R\otimes_\Lambda k(x))$ has kernel a free $R \otimes_\Lambda k(x)$-space of rank 1. Then $Ker(A)$ is a free rank 1 module over R and for every $x \in Spec(\Lambda)$ the natural map $Ker(A) \otimes_\Lambda k(x) \to Ker(A(x))$ is an isomorphism.

Proof. We proceed by induction on n. If $n = 1$, then the assertion is that A is a matrix with $A(x)$ the zero matrix for all $x \in Spec(\Lambda)$. Since R is reduced this implies that A is the zero matrix.

For the inductive step consider the system of m equations

$$\sum_i a_{ij} X_i = 0 \tag{3.14}$$

that we are trying to solve in R. If $x \in Spec(\Lambda)$ is the image of the closed point of $Spec(R)$, then $A(x)$ is not zero since $n \geq 2$. Since R is local some a_{ij} is invertible and so we can solve for the variable X_i. This gives a system of $m-1$ equations in $n-1$ variables, which again has the property that for every point $x \in Spec(\Lambda)$ the image in $R \otimes k(x)$ has a unique line of solutions. By induction we obtain the result. □

With this lemma we have proven the main proposition. □

If we apply Proposition 3.4.1 with $\mathcal{X} = \mathcal{M}_{1,1,S}$ we obtain exactly the proof of 3.2 (*ii*), i.e. of the theorem when S is reduced.

3.5. The case when S is a $\mathbb{Z}[1/2]$-scheme

We state now a technical theorem without proof. It will be useful to prove the proposition below.

Theorem 3.5.1. Let A be a noetherian local ring, $\mathfrak{a} \subset A$ an ideal of definition and $\mathcal{X}$ a proper stack. Then the functor sending a sheaf to its reductions modulo $\mathfrak{a}^n$ defines an equivalence of categories between the category of coherent sheaves on $\mathcal{X}$ and the category of compatible systems of coherent sheaves on the reductions $\mathcal{X}_n := \mathcal{X} \times_{Spec(A)} Spec(A/\mathfrak{a}^n)$.

Proof. See [Ols05], Theorem 1.4. □

We need a preliminary Proposition in order to approach our case.

Proposition 3.5.2. For any scheme S over $\mathbb{Z}[1/2]$ and any coherent $\mathcal{O}_S$-module M, the sheaf

$$R^1\pi_*(\mathcal{O}_{\mathcal{M}_{1,1,S}} \otimes_{\mathcal{O}_S} M)$$

is zero, where π is the usual projection $\pi\colon \mathcal{M}_{1,1,S} \to \mathbb{A}^1_S$.

Proof. **(Step 1)** In this step we show how to reduce the problem to the study of the case $S = k$ with k an algebraically closed field of characteristic 3.

For a sheaf of modules being zero is a local property with respect to the flat topology. So, after the flat composition $\mathcal{O}_S \to \mathcal{O}_{S,\mathfrak{p}} \to \hat{\mathcal{O}}_{S,\mathfrak{p}}$ for all $\mathfrak{p}$ point of S, we may further assume that S is the spectrum of a complete noetherian local ring A (being noetherian can be assumed because of Observation 3.3.3).

We claim that the thesis over $Spec(A/\mathfrak{m}^n)$ for all n is true. Moreover, by Theorem 3.5.1, the claim is sufficient. In fact, consider the diagram

$$\begin{tikzcd}
\mathcal{M}_{1,\,1,\,A/\mathfrak{m}^n} \arrow[r, "v_n"] \arrow[d, "\pi_n"] & \mathcal{M}_{1,\,1,\,A} \arrow[d, "\pi"] \\
\mathbb{A}^1_{A/\mathfrak{m}^n} \arrow[r, "\rho_n"] \arrow[d] & \mathbb{A}^1_A \arrow[d] \\
Spec(A/\mathfrak{m}^n) \arrow[r] & Spec(A).
\end{tikzcd}$$

The sheaf $R^1\pi_*(\mathcal{O}_{\mathcal{M}_{1,\,1,\,A}} \otimes_A M)$ corresponds to the data of $\{\rho_n^*(R^1\pi_*(\mathcal{O}_{\mathcal{M}_{1,\,1,\,A}} \otimes_A M))\}_{n\in\mathbb{N}}$. They are equal to $\{(R^1\pi_{n*}(v_n^*[\mathcal{O}_{\mathcal{M}_{1,\,1,\,A}} \otimes_A M))]\}_{n\in\mathbb{N}}$ which by our claim are equal to zero, and so the thesis.

Therefore we must prove our claim and so we reduce to the case $A/\mathfrak{m}^n$, that is a local artinian ring.

Let k be the residue field of $A/\mathfrak{m}^n$ and let $J \subset A$ be the ideal $J = \mathfrak{m}^{n-1}/\mathfrak{m}^n$. J is annihilated by the maximal ideal $\mathfrak{m}$ (so that J is a vector space). Set $A_0 := A/J = A/\mathfrak{m}^{n-1}$. Pushing forward the exact sequence

$$0 \longrightarrow J \otimes \mathcal{O}_{\mathcal{M}_{1,\,1,\,k}} \longrightarrow \mathcal{O}_{\mathcal{M}_{1,\,1,\,A}} \longrightarrow \mathcal{O}_{\mathcal{M}_{1,\,1,\,A_0}} \longrightarrow 0,$$

to $\mathbb{A}^1_A$ we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J \otimes (\pi_*\mathcal{O}_{\mathcal{M}_{1,\,1,\,k}}) \arrow[r] & \pi_*\mathcal{O}_{\mathcal{M}_{1,\,1,\,A}} \arrow[r] & \pi_*\mathcal{O}_{\mathcal{M}_{1,\,1,\,A_0}} & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{A}^1_k} \otimes J \arrow[r] \arrow[u, "a"] & \mathcal{O}_{\mathbb{A}^1_A} \arrow[r] \arrow[u, "b"] & \mathcal{O}_{\mathbb{A}^1_{A_0}} \arrow[r] \arrow[u, "c"] & 0.
\end{tikzcd}$$

Suppose to have prove the thesis for a field k, i.e. to know that a is an isomorphism. For inductive hypothesis we also know that c is an isomorphism and so b is an isomorphism too.

Hence, it follows that it suffices to consider the case $S = Spec(k)$. Furthermore, if the characteristic is not 3, the result follows from the fact the stack is tame. So it suffices to consider $S = Spec(k)$ with $char(k) = 3$, and $M = k$. We may further assume that k is algebraically closed.

(Step 2) In this step we show how to reduce the problem to a calculation on cohomology of groups.

The coherent sheaf $R^1\pi_*(\mathcal{O}_{\mathcal{M}_{1,\,1,\,k}})$ restricts to the zero sheaf on $\mathbb{A}^1_k - \{0\}$, since over this open subset of $\mathbb{A}^1_k$ the stack $\mathcal{M}_{1,\,1,\,k}$ is tame, simply because it is a μ_2-gerbe. Let $\bar{x} \to \mathcal{M}_{1,\,1,\,k}$ be a geometric point mapping to 0 in $\mathbb{A}^1_k$, and let A denote the completion of $\mathcal{O}_{\mathcal{M}_{1,\,1,\,k}\,\bar{x}}$ along the maximal ideal. Let $\Gamma_{\bar{x}}$ denote the stabilizer group scheme of $\bar{x}$, so that $\Gamma_{\bar{x}}$ acts on A. The ring of invariants $B := A^{\Gamma_{\bar{x}}}$ is equal to the completion of $\mathbb{A}^1_k$ at the origin. Let F denote the finite type B-module obtained by pulling back $R^1\pi_*(\mathcal{O}_{\mathcal{M}_{1,\,1,\,k}})$ to $Spec(B)$. Then F is equal to the cohomology group $H^1(\Gamma_{\bar{x}}, A)$. We show that this group is zero. Since F is supported on the closed point of $Spec(B)$, there exists an integer n such that $j^nF = 0$ (in fact $j \in B$ is the uniformizer defined by the standard coordinate on $\mathbb{A}^1$). Hence, to prove the proposition it suffices to show that F is j-torsion free.

For this, we use an explicit description of A and $\Gamma_{\bar{x}}$ given by the Legendre family. Let

$$V = Spec(k[\lambda][1/\lambda(\lambda-1)]),$$

and let $E_V \to V$ be the elliptic curve

$$E_V := Y^2Z = X(X-Z)(X-\lambda Z).$$

For comfort we now translate, so let μ denotes $\lambda+1$. Then the j-invariant of E_V which is $2^8\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2}$ becomes equal to $\frac{\mu^6}{\mu^4-1}$ (recall that char(k)=3). The map $V \to \mathcal{M}_{1,\,1,\,k}$ defined by E_V is étale, so it defines an isomorphism $A \cong k[[u]]$. We have seen in Theorem 2.4.1 that the group $\Gamma_{\bar{x}}$ sits in an exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \Gamma_{\bar{x}} \longrightarrow S_3 \longrightarrow 1.$$

To be more precise, the isomorphisms of a curve of the form $y^2 = x^3 + a_4x^2 + a_6$ over an algebraically closed field of characteristic 3 are of the form

$$x = u^2x' + r$$
$$y = u^3y',$$

with u and r satisfying $u^4 = 1$ and $r^3 + a_4r + a_6(1-u^6) = 0$. The pair $(u, r) = (-1, 0)$ which generates the subgroup of $\Gamma_{\bar{x}}$ corresponding to $\{\pm 1\}$, is the classic involution of elliptic curves, and it acts trivially on $k[[\mu]]$ as we have already seen in Lemma 3.1.1.

To understand the action of S_3 we use a different argument. Two elliptic curves over the Legendre family are isomorphic if and only if $\lambda \in \{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\}$. Given $\lambda \in k$, these six values are all different except that for $\lambda = -1$ (since we are in $char(k) = 3$, then $-1 = 2 = \frac{1}{2}$). Observe that the group S_3 acts on V. To be more explicit, it defines a morphism of groups

$$S_3 \hookrightarrow Aut(V)$$
$$(23) \mapsto [\lambda \mapsto 1-\lambda]$$
$$(312) \mapsto [\lambda \mapsto \frac{\lambda-1}{\lambda}].$$

This action lifts to isomorphisms of elliptic curves that, as already observed, in the case $\lambda = -1$ (which we recall coincides with $j = 0$) are automorphisms. In particular, for uniqueness, we have found the subgroup of $\Gamma_{\bar{x}}$ isomorphic to S_3 and its action on V. After the translation $\mu = \lambda + 1$ we obtain $[\lambda \mapsto 1-\lambda] \mapsto [\mu \mapsto -\mu] =: \alpha$ and $[\lambda \mapsto \frac{\lambda-1}{\lambda}] \mapsto [\mu \mapsto \frac{\mu}{1-\mu}] =: \beta$.

Recapping: we have proved that the action of $\Gamma_{\bar{x}}$ on $A \cong k[[u]]$ factors through the action of S_3 given by the two automorphisms

$$\alpha\colon \mu \mapsto -\mu$$

and

$$\beta\colon \mu \mapsto \frac{\mu}{1-\mu} = \mu(1 + \mu + \mu^2 + \ldots).$$

If we consider the Lyndon/Hochschild-Serre Spectral Sequence associated, see Theorem 6.8.2 in [Wei95] for more details, we obtain:

$$E_2^{pq} = H^p(S_3, H^q(\{\pm 1\}, A)) \implies H^{p+q}(\Gamma_{\bar{x}}, A).$$

and the exact sequence in the lower degree filtration:

$$0 \longrightarrow H^1(\Gamma_{\bar{x}}/\{\pm 1\}, A^{\{\pm 1\}}) \longrightarrow H^1(\Gamma_{\bar{x}}, A) \longrightarrow H^1(\{\pm 1\}, A)^{\Gamma_{\bar{x}}/\{\pm 1\}} \longrightarrow \ldots.$$

This sequence coincides with

$$0 \longrightarrow H^1(S_3, A) \longrightarrow H^1(\Gamma_{\bar{x}}, A) \longrightarrow 0 \longrightarrow \ldots.$$

since the action of $\{\pm 1\}$ is trivial and $H^1(\{\pm 1\}, A)) = Hom(A, \mathbb{Z}/2\mathbb{Z}) = 0$. Hence $H^1(\Gamma_{\bar{x}}, A) \cong H^1(S_3, A)$.

(Step 3) In this final step, we prove that $H^1(S_3, A) = 0$.

By standard facts about group cohomology, an element in $H^1(S_3, A)$ can be represented by a set map $\xi\colon S_3 \to k[[\mu]]$ (written $\sigma \mapsto \xi_\sigma$) such that for $\sigma, \tau \in S_3$ we have (recall the action is a right action)

$$\xi_{\sigma\tau} = \xi_\sigma^\tau + \xi_\tau. \tag{3.15}$$

The class of ξ is trivial if there exists an element $g \in k[[\mu]]$ such that $\xi_\sigma = g^\sigma - g$ for all $\sigma \in S_3$. Note that equation 3.15 implies that it suffices to check the equality $\xi_\sigma = g^\sigma - g$ for a set of generators of S_3.

If ξ represents a class in $H^1(S_3, A)$ annihilated by j, then there exists an element $g \in k[[\mu]]$ such that

$$\frac{\mu^6}{\mu^4 - 1}\xi_\sigma = g^\sigma - g \tag{3.16}$$

for all $\sigma \in S_3$. To prove that $H^1(S_3, A)$ is j-torsion free, it therefore suffices to show that for such a ξ we can choose g to have μ-adic valuation ≥ 6 (since A is j-torsion free). In this way ξ is a multiple of $\frac{\mu^6}{\mu^4-1}$ and so its class in $H^1(S_3, A)$ is zero.

For this, first of all note that we can assume without loss of generality that g has no constant term, and then write

$$g = a_1\mu + a_2\mu^2 + a_3\mu^3 + a_4\mu^4 + a_5\mu^5 + g_{\geq 6}, \tag{3.17}$$

where $g_{\geq 6}$ has μ-adic valuation ≥ 6.

We have that

$$\frac{\mu^6}{\mu^4 - 1}\xi_\alpha = 2a_1\mu + 2a_3\mu^3 + 2a_5\mu^5 + (g^\alpha_{\geq 6} - g_{\geq 6}),$$

which implies that $a_1 = a_3 = a_5 = 0$.

Instead we have that

$$\frac{\mu^6}{\mu^4 - 1}\xi_\beta = a_2(\frac{\mu}{1-\mu})^2 + a_4(\frac{\mu}{1-\mu})^4 - a_2\mu^2 - a_4\mu^4 + (\text{higher order terms})$$

and hence

$$\begin{aligned}\frac{(1-\mu)^4\mu^6}{1-\mu^4}\xi_\beta = {} & a_2\mu^2(1-\mu)^2 - a_2\mu^2(1-\mu)^4 + \\ & + a_4\mu^4 - a_4\mu^4(1-\mu)^4 + \\ & + (\text{higher order terms}).\end{aligned}$$

We observe that on the right hand side the term with μ^2 disappears and the lowest order term is the one with μ^3, in particular $-2a_2\mu^3 + 4a_2\mu^3 = 2a_2\mu^3$, and therefore $a_2 = 2$. Replacing a_2 with zero, the lowest order term om the right hand side becomes $a_4\mu^5$ and at the same way $a_4 = 0$ as desired. This completes the proof.

□

We are now ready to give a proof of the Theorem 3.0.1, part (i).

Proof. Let S be a connected $\mathbb{Z}[1/2]$-scheme. We need to show that if L is a line bundle on $\mathcal{M}_{1,1,S}$ such that for any field-valued point $s \in S$ the fiber L_s on $\mathcal{M}_{1,1,S}$ descends to $\mathbb{A}^1_s$, then L descends to $\mathbb{A}^1_S$. An other way to say this is, following the notation of Section 3.3, that we need to show that S has the ♣ property.

By Lemma 3.3.2 and Observation 3.3.3, it suffices to prove the thesis for an affine connected noetherian $\mathbb{Z}[1/2]$-scheme $S = Spec(\Lambda)$. Let $J \subset \Lambda$ denote the nilradical. By the reduced case already treated in the previous section, it suffices to show inductively that if the result holds for Λ/J^r then it also holds for Λ/J^{r+1}. We are in the following situation:

$$\begin{tikzcd}
\mathcal{M}_{1,1,\Lambda/J^r} \arrow[r, "\pi"] \arrow[d, "g"] & \mathbb{A}^1_{\Lambda/J^r} \arrow[r] \arrow[d] & Spec(\Lambda/J^r) \arrow[d] \\
\mathcal{M}_{1,1,\Lambda/J^{r+1}} \arrow[r, "\pi"] & \mathbb{A}^1_{\Lambda/J^{r+1}} \arrow[r] & Spec(\Lambda/J^{r+1})
\end{tikzcd}$$

Let $\mathcal{L}$ denote a line bundle on $\mathcal{M}_{1,1,\Lambda/J^{r+1}}$ such that for any field-valued point $s \in S$ the fiber $\mathcal{L}_s$ descend to $\mathbb{A}^1_{k(s)}$. We want to show that there exists a line bundle L on $\mathbb{A}^1_{\Lambda/J^{r+1}}$ whose pull back is $\mathcal{L}$. By inductive hypothesis, there is a line bundle L_0 on $\mathbb{A}^1_{\Lambda/J^r}$ such that $\pi^*L_0 = g^*\mathcal{L}$ on $\mathcal{M}_{1,1,\Lambda/J^r}$. Our claim is that $\mathcal{L}$ is pulled back from a lifting of L_0 to $\mathbb{A}^1_{\Lambda/J^{r+1}}$. Observe that $Spec(\Lambda/J^r) \to Spec(\Lambda/J^{r+1})$ is a closed embedding defined by the square zero ideal $I = J^r/J^{r+1}$. Since the diagram is cartesian the other vertical arrows are closed embedding defined by a square zero ideal too (in the sense of stacks). So the hypothesis of Theorem 6.4 of [Har10] are satisfied and we can apply it to conclude. Since $H^2(\mathbb{A}^1_{\Lambda/J^r}, J^r/J^{r+1} \otimes \mathcal{O}_{\mathbb{A}^1_{\Lambda/J^r}})$ is zero, a lifting for L_0 exists, and it is unique since the group $H^1(\mathbb{A}^1_{\Lambda/J^r}, J^r/J^{r+1} \otimes \mathcal{O}_{\mathbb{A}^1_{\Lambda/J^r}}) = 0$ acts transitively on the set of all isomorphism classes of deformations $\bar{L}$ of L_0 over $\mathbb{A}^1_{\Lambda/J^{r+1}}$. We denote with L this lifting. By construction, π^*L is a lifting of $g^*\mathcal{L}$. If we prove that it is isomorphic to $\mathcal{L}$ we are done. Again, it suffices to prove that $H^1(\mathcal{M}_{1,1,\Lambda/J^r}, J^r/J^{r+1} \otimes \mathcal{O}_{\mathcal{M}_{1,1,\Lambda/J^r}})$ is zero. Since $\mathbb{A}^1_{\Lambda/J^r}$ is affine, the group $H^1(\mathcal{M}_{1,1,\Lambda/J^r}, J^r/J^{r+1} \otimes \mathcal{O}_{\mathcal{M}_{1,1,\Lambda/J^r}})$ is zero if and only if the sheaf $R^1\pi_*(J^r/J^{r+1} \otimes \mathcal{O}_{\mathcal{M}_{1,1,\Lambda/J^r}})$ is zero on $\mathbb{A}^1_\Lambda$, which follows from Proposition 3.5.2. □

3.6. Computations in characteristic 2

Proposition 3.6.1. Let k be a field of characteristic 2, and let $\pi\colon \bar{\mathcal{M}}_{1,1,k} \to \mathbb{P}^1_k$ be the morphism defined by the j-invariant. Then $R^1\pi_*\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}}$ is a line bundle on $\mathbb{P}^1_k$ of negative degree.

Proof. First of all, suppose without loss of generality that k is algebraically closed. We divide the proof in steps.

(Step 1). In this step we prove that in general, if $f\colon \mathcal{G} \to X$ is a $\mathbb{Z}/2$-gerbe in characteristic 2, the sheaf $R^1f_*\mathcal{O}_{\mathcal{G}}$ is locally free of rank 1, and in fact it is canonically trivialized.

This can be seen as follows. There is an étale cover $\{U_i\}_i$ which trivializes the gerbe, i.e. if U is an open of the cover, $\mathcal{G}_U := \mathcal{G} \times_X U \cong \mathbb{B}(\mathbb{Z}/2) \times U \cong [U/(\mathbb{Z}/2)]$ and the action $\rho\colon G \to U$ is trivial. Let $f_U\colon \mathcal{G}_U \to U$ the morphism into the coarse moduli space. Suppose without loss of generality U to be affine, say $U = Spec(R)$. Clearly by the definition of cohomology of group, it follows that $R^1f_U^*\mathcal{O}_{\mathcal{G}_U} = H^1(\mathbb{Z}/2, R)$. By definition, an element in $H^1(\mathbb{Z}/2, R)$ is represented by a set map $\xi\colon \mathbb{Z}/2 \to R$ (written $\sigma \mapsto \xi_\sigma$) such that for $\sigma, \tau \in \mathbb{Z}/2$ we have $\xi_{\sigma\tau} = \xi_\sigma^\tau + \xi_\tau$. But the action is trivial, thus we have $\xi_{\sigma\tau} = \xi_\sigma + \xi_\tau$, hence $\xi \in Hom_{Grp}(\mathbb{Z}/2, R) \cong R$. So we have $R^1f_U^*\mathcal{O}_{G_U} \cong \mathcal{O}_U$.

Similarly, consider the locally constant sheaf $\mathbb{Z}/2$ on $\mathcal{G}$. The sheaf $R^1f_*(\mathbb{Z}/2)$ is locally isomorphic to $\mathbb{Z}/2$. In fact, if we take a trivializing étale open U for the gerbe as above, $R^1f_*(\mathbb{Z}/2)|_U \cong R^1f|_{U,*}((\mathbb{Z}/2)_U)$ for base change Theorem, see [Sta22] Tag [0EYU]. In the same way as before, $R^1f|_{U,*}((\mathbb{Z}/2)_U) = H^1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2Z$.

Since we are in characteristic 2, we have a natural map $\mathbb{Z}/2 \to \mathcal{O}_{\mathcal{G}}$, which leads to $f_*\mathbb{Z}/2\mathbb{Z} \to f_*\mathcal{O}_{\mathcal{G}}$. Again, since we are in characteristic 2, we can tensor with $\mathcal{O}_X$ to obtain a morphism of $\mathcal{O}_X$-sheaves $\Psi\colon f_*\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}/2} \mathcal{O}_X \to f_*\mathcal{O}_{\mathcal{G}}$. This map is an isomorphism: if we restrict Ψ to an appropriate étale cover $\{U_i\}_i$, it is exactly what we have verified with the initial discussion.

Finally, the sheaf $f_*(\mathbb{Z}/2) \cong \mathbb{Z}/2$. Actually, locally we have seen it works. Moreover, the transition maps between the trivializing opens must be the identity since a group of order 2 admits just the trivial automorphism. This concludes the proof of our initial claim.

Recall that in the case of $\mathcal{G}_U = U \times \mathbf{B}(\mathbb{Z}/2)$ and $U = Spec(R)$ we have

$$H^1(\mathcal{G}, \mathcal{O}_{\mathcal{G}}) \cong Hom_{\mathrm{Grp}}(\mathbb{Z}/2, R)$$

and the trivialization is given by the homomorphism sending $1 \in \mathbb{Z}/2$ to $1 \in R$.

(Step 2). In this step we prove that the sheaf $R^1\pi_*(\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}})$ is locally free of rank 1 on $\mathbb{P}^1_k$.

Let $\mathcal{U}_\infty, \mathcal{U}_0 \subset \bar{\mathcal{M}}_{1,1,k}$ denote the open substacks of $\bar{\mathcal{M}}_{1,1,k}$ obtained pulling back the affine standard open covers $U_\infty, U_0 \subset \mathbb{P}^1_k$ (respectively the complement of $j = 0$ and the complement of $j = \infty$).

The stack $\mathcal{U}_\infty$ is a $\mathbb{Z}/(2)$-gerbe over U_∞ (see Proposition 2.4.1), hence for the argument of Step 1., we deduce that $R^1\pi|_{\mathcal{U}_\infty}(\mathcal{O}_{\mathcal{U}_\infty})$ is locally free of rank 1. Moreover, $R^1\pi_*(\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}})$ is coherent. In fact, if $f\colon V \to \bar{\mathcal{M}}_{1,1,k}$ is a smooth atlas, $R^1\pi_*(\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}})$ is a sub-sheaf of $R^1(\pi \circ f)_*(\mathcal{O}_V)$. However, $R^1(\pi \circ f)_*(\mathcal{O}_V)$ is coherent since the map $\pi \circ f$ is proper and $\mathbb{P}^1_S$ locally noetherian (we are applying Proposition 30.19.1 of [Sta22] Tag [02O3]), and so $R^1(\pi \circ f)*(\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}})$ is coherent too. Since $\mathbb{P}^1_k$ is a smooth curve, if we prove that $R^1(\pi \circ f)*(\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}})$ is torsion-free then it is necessarily free. Furthermore, the rank is 1 because so is that of $R^1\pi|_{\mathcal{U}_\infty}(\mathcal{O}_{\mathcal{U}_\infty})$.

Furthermore, the only issue is at the point $j = 0$. Since the formation of cohomology commutes with flat base change, it suffices to show that

$$H^1(\bar{\mathcal{M}}_{1,1,k} \times_{\mathbb{P}^1} Spec(k[[j]]), \mathcal{O}_{\bar{\mathcal{M}}_{1,1,k} \times_{\mathbb{P}^1} Spec(k[[j]])}) \tag{3.18}$$

is j-torsion-free.

For this we use the Hesse presentation of $\bar{\mathcal{M}}_{1,1,k}$. For the details about the fact that it is indeed a presentation for the stack and a direct computation we urge the reader to see the Appendix A in the article [Shi19] where everything is excellently exposed. Let

$$V = Spec(k[\mu, \omega][1/(\mu^3 - 1)]/(\omega^2 + \omega + 1)),$$

and let $E_V \to V$ be the elliptic curve given by the equation

$$X^3 + Y^3 + Z^3 = \mu XYZ.$$

The elliptic curve has a basis for its three-torsion group given by the points $[1,0,-1]$ and $[-1,\omega,0]$, and identity section $[1,-1,0]$. In fact, this is the universal elliptic curve with full level three structure. The j-invariant of E_V is $\frac{\mu^{12}}{(\mu^3-1)^3}$ since we are in characteristic 2. In particular, the fiber over $j=0$ is the curve $X^3+Y^3+Z^3=0$.

Changing the choice of a basis for the 3-torsion subgroup, defines an action of $GL_2(\mathbb{Z}/3)$ on V such that $\bar{\mathcal{M}}_{1,1,k} \cong [V/GL_2(\mathbb{Z}/3)]$. A calculation shows that this action is described as follows:

$$\begin{cases} (\mu,\omega)\star\left(\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right) = (\omega\mu,\omega) \\ (\mu,\omega)\star\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) = (\frac{\mu+2}{\mu-1},\omega) \\ (\mu,\omega)\star\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) = (\mu,\omega^2) \end{cases}$$

Putting this together one finds that

$$\bar{\mathcal{M}}_{1,1,k}\times_{\mathbb{P}^1} Spec(k[[j]]) \cong [Spec(k[[\mu]]/SL_2(\mathbb{Z}/3)], \tag{3.19}$$

where $\alpha = \left(\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right)$ acts by $\mu\mapsto\zeta_3\mu$ (for some fixed primitive cube root of unity ζ_3) and $\beta=\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ acts by $\mu\mapsto\frac{\mu}{\mu-1}$.

As in the proof of Proposition 3.5.2, an element of 3.18 is given by a set map $\xi\colon SL_2(\mathbb{Z}/3)\to k[[\mu]]$ (written $\sigma\mapsto\xi_\sigma$) such that for any two elements $\sigma,\tau\in SL_2(\mathbb{Z}/3)$ we have

$$\xi_{\sigma\tau} = \xi_\sigma^\tau + \xi_\tau,$$

and the class of ξ is trivial if there exists an element $g\in k[[\mu]]$ such that for every σ we have $\xi_\sigma = g^\sigma - g$.

Now, if 3.18 has j-torsion, there exists a set map ξ as above and an element $g\in k[[\mu]]$ such that for all σ we have

$$\frac{\mu^{12}}{\mu^3-1}\xi_\sigma = g^\sigma - g.$$

To prove that 3.18 is torsion-free, it suffices to show that we can choose g to be divisible by μ^{12}.

We can without loss of generality assume that g has no constant term. Write

$$g = a_1\mu + a_2\mu^2 + \ldots + a_{11}\mu^{11} + g_{\geq 12}.$$

Then $g^\sigma - g$ has μ-adic valuation ≥ 12 for all σ, so this is in particular for $\sigma=\alpha$. However, $g^\alpha - g$ $= a_1(\zeta_3-1)\mu + a_2(\zeta_3^2-1)\mu^2 + a_4(\zeta_4-1)\mu^4+\ldots$ and this implies that all the the coefficients a_i but a_3, a_6 and a_9 are zero. So

$$g = a_3\mu^3 + a_6\mu^6 + a_9\mu^9 + g_{\geq 12}.$$

Similarly, $g^\beta - g$ has μ-adic valuation ≥ 12. Looking at the coefficient of μ^4 in $g^\beta - g$ one sees that $a_3=0$. Then looking at the coefficient of μ^7 one sees that $a_6=0$, and finally looking at the coefficient of μ^{10} one sees that $a_9=0$. The calculation is omitted, since it is substantially equal to the case of β in the Proposition 3.5.2.

This concludes the proof that $R^1\pi_*\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}}$ is locally free of rank 1 on $\mathbb{P}^1_k$.

(Step 3). The final step consists in showing that the degree of $R^1\pi_*\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}}$ is negative.

Let M denote the cohomology group 3.18 (a $k[[j]]$-module) and let M_η denote the pullback $M\otimes_{k[[j]]}k[[j]][1/j]$. Let $e_\infty\in M_\eta$ denote the basis element defined by the canonical trivialization of $R^1\pi_*\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}}$ over $\mathcal{U}_\infty$. The lattice $M\subset M_\eta$ defines a valuation ν on M_η and, for the standard arguments about the sheaves $\mathcal{O}(i)$'s of $\mathbb{P}^1_k$, the thesis is equivalent to show that $\nu(e_\infty)<0$. Equivalently, we have to show that for any element $m\in M$, if we write $m = he_\infty$ in M_η then the j-adic valuation of h is positive.

For this again we use the presentation 3.19. An element $m\in M$ is then represented by a map $\xi\colon SL_2(\mathbb{Z}/2)\to k[[\mu]]$. The corresponding element in M_η can be described in terms of the basis e_∞ as follows. First of all, the element $\xi_{\beta^2}\in k[[\mu]]$ is $SL_2(\mathbb{Z}/2)$-invariant, since for any other element σ we have

$$\xi_{\beta^2}^\sigma + \xi_\sigma = \xi_{\beta^2\sigma} = \xi_{\sigma\beta^2} = \xi_\sigma^{\beta^2} + \xi_{\beta^2} = \xi_\sigma + \xi_{\beta^2}.$$

The second equality works since $\beta^2 = \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$ commutes with every element of $SL_2(\mathbb{Z}/2)$, and the last equality since β^2 acts on μ by $\mu\mapsto\frac{\mu}{\mu-1}\mapsto\frac{\mu}{1-2\mu}=\mu$ (in other words β^2 acts trivially on $k[[\mu]]$).

Therefore, ξ_{β^2} is actually an element in $k[[j]]$. The image of ξ in $M_\eta \cong Hom(\mathbb{Z}/2, k[[j]][1/j])$ is then equal to the homomorphism

$$\begin{aligned} \mathbb{Z}/2 &\to k[[j]][1/j] \\ 1 &\mapsto \xi_{\beta^2}. \end{aligned}$$

The class e_∞ corresponds to the homomorphism sending 1 to 1, so we have to show that the j-adic valuation of ξ_{β^2} is positive. For this, let $f = \xi_\beta$. Then

$$\xi_{\beta^2} = f^\beta + f = f(\mu(1+\mu+\mu^2+\ldots)) + f(\mu).$$

Since we are in characteristic 2, it follows that the μ-adic valuation of ξ_{β^2} is at least 2, and therefore the j-adic valuation of ξ_{β^2} is positive, which proves the thesis.

□

Observe that in particular $H^0(\mathbb{P}^1_k, R^1\pi^*\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}}) = 0$. From this fact we can prove the following fact.

Corollary 3.6.2. For any field k, we have $H^1(\bar{\mathcal{M}}_{1,1,k}, \mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}}) = 0$.

Proof. We have that $R^1\pi_*\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}} = 0$ when $char(k) \neq 2$. Actually when $char(k) \neq 2, 3$ it follows from the fact that the stack is tame. When $char(k) = 3$ it follows from Proposition 3.5.2. It follows that

$$H^0(\mathbb{P}^1_k, R^1\pi_*\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}}) = 0$$

in all characteristics. From the Leray Spectral Sequence, see Theorem 5.8.6 in [Wei95], which is a particular case of the Grothendieck Spectral Sequence, see Theorem 5.8.3 in [Wei95] we obtain

$$0 \longrightarrow H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \longrightarrow H^1(\bar{\mathcal{M}}_{1,1,k}, \mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}}) \longrightarrow H^0(\mathbb{P}^1_k, R^1\pi_*\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}})$$

and from the fact that the first and the third term are equal to zero, it follows that

$$H^1(\bar{\mathcal{M}}_{1,1,k}, \mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}}) = 0.$$

□

Counter Example 3.6.3. We want to prove that Theorem 3.0.1 fails in the case in which $S = Spec(k[\varepsilon]/(\varepsilon^2))$ with $char(k) = 2$.

Note that, as we have seen in Proposition 3.6.1, if $char(k) = 2$, then the restriction of $R^1\pi_*\mathcal{O}_{\bar{\mathcal{M}}_{1,1,k}}$ to $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ is nonzero. From the Leray Spectral Sequence, applied as in the previous corollary, we obtain

$$0 = H^1(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) \xrightarrow{f} H^1(\mathcal{M}_{1,1,k}, \mathcal{O}_{\mathcal{M}_{1,1,k}}) \longrightarrow H^0(\mathbb{A}^1_k, R^1\pi_*\mathcal{O}_{\mathcal{M}_{1,1,k}}) \neq 0.$$

Then f is not an isomorphism.

Since the group $H^1(\mathcal{M}_{1,1,k}, \mathcal{O}_{\mathcal{M}_{1,1,k}})$ classifies deformations of the structure sheaf to $\mathcal{M}_{1,1,k[\varepsilon]/(\varepsilon^2)}$, see [Har10] Proposition 2.5 this implies that there are line bundles L on $\mathcal{M}_{1,1,k[\varepsilon]/(\varepsilon^2)}$ such that for every point $s \to k[\varepsilon]/(\varepsilon^2)$, L_s descends to $\mathbb{A}^1_{k(s)}$, but are not trivial. These represents the counter examples.

APPENDIX A

Tame Stacks

In characteristic 0 Deligne-Mumford stacks own some nice properties that make them easy to study. In particular, let $\mathcal{M}$ be a Deligne-Mumford stack and $\mathcal{M} \to M$ its coarse moduli space. Étale locally on M, we can present $\mathcal{M}$ as a quotient stack 1.8.4. From this fact it follows that, in characteristic 0, the formation of M commutes with arbitrary base change. A key property in characteristic 0 is that the pushforward functor $(Qcoh(\mathcal{M})) \to (Qcoh(M))$ is exact.

In characteristic $p > 0$ these results are not true. The main problems that one can encounter appear when one consider algebraic stack with finite ramified diagonal (hence not Deligne-Mumford). Moreover, even limiting oneself to consider Deligne-Mumford stacks in characteristic $p > 0$, some desired properties don't hold, such as flatness of moduli spaces. This happens when the order of the stabilizers are divisible by the characteristic.

For these reasons, *tame stacks* are studied. These stacks are a good substitute of the Deligne-Mumford stacks in positive and mixed characteristics. Their defining property is precisely the key property described above: an algebraic stack with finite diagonal is tame if and only if the pushforward functor $(Qcoh(\mathcal{M})) \to (Qcoh(M))$ is exact. The pleasant consequence of this property is that other desirable properties follow as corollaries. For example, for tame stacks an analogous of Theorem 1.8.4 holds and the formation of coarse moduli space commutes with arbitrary base change.

The study of tame stack goes beyond the purpose of the thesis; therefore in this section we just present (without proof) the facts we need for our applications to the stack of elliptic curves. For more details we urge the reader to view the original article [AOV08].

Throughout the section all schemes are assumed to be quasi-separated. Moreover, recall that we consider only flat, finite and finitely presented affine group schemes over a scheme S.

We start by defining linearly reductive group schemes. We need to define them because they are indeed exactly the automorphism groups of the geometric points of tame stacks.

Definition A.0.1. A group scheme $G \to S$ is *linearly reductive* if the functor

$$(Qcoh^G(S)) \to (Qcoh(S))$$
$$F \mapsto F^G$$

is exact.

Observation A.0.2. If k is a field, since the category of quasi-coherent sheaves on $Spec(k)$ with an action of G is equivalent to the category of finite-dimensional representations of G (Proposition

1.10.7), hence a finite group scheme over a field is linearly reductive if and only if the functor $V \mapsto V^G$, from finite-dimensional representations of G to vector spaces, is exact.

Another way to state this condition is that every finite-dimensional representation of G is a sum of irreducible representations.

We state some nice properties about linearly reductive group schemes.

Proposition A.0.3. Let $S' \to S$ be a morphism of schemes, $G \to S$ a group scheme, $G' := S' \times_S G$.

1. If $G \to S$ is linearly reductive, then $G' \to S'$ is linearly reductive.
2. If $G' \to S'$ is linearly reductive and $S' \to S$ is flat and surjective, then $G \to S$ is linearly reductive.

Proof. See [AOV08], Proposition 2.6. □

Proposition A.0.4. The class of linearly reductive schemes is closed under taking

1. subgroups schemes,
2. quotients, and
3. extensions.

Recall that datum of flat, finite and finitely presented affine group scheme $G = \mathit{Spec}(A)$ over a scheme $S = \mathit{Spec}(R)$ is equivalent to the datum of a finite flat Hopf algebra A over R. In particular, we have morphisms

$$\begin{aligned} m\colon A \otimes A &\to A, \\ c\colon A &\to A \otimes A, \\ R &\to A, \\ e\colon A &\to R, \\ i\colon A, &\to A, \end{aligned}$$

where c, e and i are the data of the definition of R-Hopf algebra, $R \to A$ is the structural morphism and m is the multiplication map. Dualizing A and the morphisms we obtain

$$\begin{aligned} m^\vee\colon A^\vee &\to A^\vee \otimes A^\vee, \\ c\colon A^\vee \otimes A^\vee &\to A^\vee, \\ A^\vee &\to R, \\ e^\vee\colon R &\to A^\vee, \\ i^\vee\colon A^\vee, &\to A^\vee. \end{aligned}$$

With these morphisms, $A^\vee$ becomes an R-Hopf algebra, with $A^\vee$ finite and flat over R, and $G^\vee := \mathit{Spec}(A^\vee)$ is called Cartier dual of G.

Definition A.0.5 (Diagonalizable group)**.** We will say that a finite group scheme $\Delta \to S$ is *diagonalizable* if it is abelian and its Cartier dual is constant.

We say that it is *locally diagonalizable* if locally in the fpqc topology is diagonalizable.

Definition A.0.6 (Tame group)**.** We say that a finite étale group scheme $H \to S$ is *tame* if its degree is prime to all residue characteristics.

Definition A.0.7. A group scheme $\pi\colon G \to S$ is *well-split* if it is isomorphic to a semidirect product $H \ltimes \Delta$, where H is constant and tame and Δ is diagonalizable.

It is *locally well-split* if there is an fpqc cover $\{S_i \to S\}$, such that the group scheme $S_i \times_S G \to S_i$ is well-split for each i.

Proposition A.0.8. Every locally-split group scheme is linearly reductive.

Proof. See [AOV08], Proposition 2.10. □

Observe that in characteristic 0 every finite flat group scheme is linearly reductive: actually, it is étale and tame, hence locally constant, hence locally well-split.

In positive characteristic the things are more complicated.

The viceversa of the previous proposition is not true, but it is true in a particular case:

Proposition A.0.9. Let k be a field, $G \to Spec(k)$ a finite group scheme. Then G is linearly reductive if an only if it is locally well-split.

Proof. See [AOV08], Proposition 2.13. □

After this brief introduction we are ready to give the definition of tame stack, a notion that we will use for Lemma 3.1.2 in chapter 4. Let S be a scheme, $\mathcal{M} \to S$ a locally finitely presented algebraic stack over S, with finite diagonal. Hence by the Theorem 1.8.2 we know that it has the coarse moduli space $\rho\colon \mathcal{M} \to M$.

Definition A.0.10 (Tame Stack)**.** The stack $\mathcal{M}$ is said to be *tame* if the functor

$$\rho^*\colon (Qcoh(\mathcal{M})) \to (Qcoh(M))$$

is exact.

For example, when $G \to S$ is a finite flat group scheme, then the moduli space of $\mathbf{B}_S G \to S$ is S itself; so $\mathbf{B}_S G$ is tame if and only if G is linearly reductive.

The main theorem about tame stacks is the following.

Theorem A.0.11. The following conditions are equivalent:

1. $\mathcal{M}$ is tame.
2. If k is an algebraically closed field with a morphism $Spec(k) \to S$ and ξ is an object of $\mathcal{M}(Spec(k))$, then the automorphism group scheme $\underline{Aut}_k(\xi) \to Spec(k)$ is linearly reductive.
3. There exists an fppf cover $M' \to M$, a linearly reductive group scheme $G \to M'$ acting on a finite and finitely presente scheme $U \to M'$, together with an isomorphism
$$\mathcal{M} \times_M M' \cong [U/G]$$
of algebraic stacks over M'.
4. Same as the 3., but $M' \to M$ is assumed to be étale and surjective.

Proof. See [AOV08], Theorem 3.2. □

Corollary A.0.12. Let $\mathcal{M}$ be a tame stack over a scheme S and let $\mathcal{M} \to M$ be its moduli space.

1. If $M' \to M$ is a morphism of algebraic spaces, then the moduli space of $M' \times_M \mathcal{M}$ is M'.
2. If $\mathcal{M}$ is flat over S, then M is also flat over S.

Proof. See [AOV08], Corollary 3.3 □

We conclude stating some immediate corollaries.

Corollary A.0.13. If $\mathcal{M} \to S$ is a tame stack and $S' \to S$ is a morphism of schemes, then $S' \times_S \mathcal{M}$ is a tame stack over S'.

Corollary A.0.14. The stack $\mathcal{M} \to S$ is tame if and only if for any morphism $Spec(k) \to S$, where k is an algebraically closed field, the geometric fiber $Spec(k) \times_S \mathcal{M}$ is tame.

APPENDIX B

Excellent and Nagata Rings

We recall Grothendieck's notion of *excellent rings* and their relation with *Nagata rings*. These concepts go beyond the scope of the thesis. The reason they are presented is to provide, at the end of this section, a proof of Theorem 3.2.2.

Definition B.0.1. Let R be a domain with quotient field $K(R)$. R is $N-2$ or *Japanese* if for any finite extension $K(R) \subset L$ of fields the integral closure $\bar{R}^L$ is finite over R.

Definition B.0.2. Let R be a ring.

- R is a *Nagata ring* if R is noetherian and for every prime ideal $\mathfrak{p}$ the domain $R/\mathfrak{p}$ is $N-2$.
- Let S be a ring. $R \to S$ is *essentially of finite type* if S is the localization of an R-algebra of finite type.

Lemma B.0.3. Let R be a Nagata ring. Let $R \to S$ be essentially of finite type with S reduced. Then the integral closure $\bar{R}^S$ is finite over R.

Proof. See [Sta22], Tag [$03GH$]. □

Definition B.0.4. Let R be a ring.

- R is a *G- ring* if R is noetherian and for every prime $\mathfrak{p}$ of R the ring map $R_\mathfrak{p} \to \hat{R}_\mathfrak{p}$ is regular.
- R is a *J-2* if it is noetherian and for any finite type R-algebra S, the set

  $$Reg(Spec(S)) := \{\mathfrak{p} \in Spec(S) \mid R_\mathfrak{p} \text{is a regular local ring}\}$$

  is open.
- R is *universally catenary* if every R-algebra of finite type is catenary.
- R is *quasi-excellent* if R is noetherian, a $G-2$ ring and $J-2$.
- R is *excellent* if R is quasi-excellent and universally catenary.

Lemma B.0.5. The following types of rings are excellent:

- fields,

- noetherian complete local rings,
- $\mathbb{Z}$,
- Dedekind domains with fraction field of characteristic zero,
- finite type ring extensions of any of the above.

Proof. See [Sta22], Tag [07QW]. $\square$

Lemma B.0.6. A quasi-excellent ring is Nagata.

Proof. See [Sta22], Tag [07QV]. $\square$

After these preliminaries we are now ready to carry a proof of the Theorem 3.2.2, that we state again now:

Theorem B.0.7. Let Λ be a normal domain, then it is the filtered colimit of a system consisting of finite type and normal subrings.

Proof. By B.0.5 a finite type ring over $\mathbb{Z}$ is excellent. Therefore, any finitely generated subring R_α of R is excellent and so Nagata by B.0.6. Since R is normal, then the integral closure $\bar{R_\alpha}^{K(R_\alpha)}$, where $K(R_\alpha)$ is the quotient field of R_α, is contained in R. Since R_α is Nagata and $K(R_\alpha)$ is the localization of R_α by the prime $\mathfrak{p} = (0)$, then the Lemma B.0.3 applies and $\bar{R_\alpha}^{K(R_\alpha)}$ is finite over R_α. Therefore $\{\bar{R_\alpha}^{K(R_\alpha)}\}$ is a filtering system of finite type normal subrings of R. $\square$

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